Abstract

The paper deals with the concept of coherent risk measure, in the sense of Artzner, Delbaen, Eber and Heath (1999), but assumes a dynamic setting in which risks (actuarial problems) or pay-offs (financial problems) reflect their stochastic behavior.

First, the definition of a dynamic coherent measure is introduced and some relations with its static version are given. Coherent measures are characterized by martingale-like representations, as well as by Choquet integrals of probability distortion operators.

Using different information updating rules (e.g. Bayes or Dempster-Shafer) we define three distinct families of dynamic coherent measures. In particular, we extend Value at Risk (VaR) and Tail Conditional Expectation (TCE) to a dynamic setting.

1 Introduction

Measuring the level of exposure to market risk is of interest in financial analysis. Market risk rises from fluctuations in the price of assets, goods, interest rates or
other related variables. Regulators often use risk measures to assess the impact of extreme scenarios on a position. These measures are then used to determine the additional capital required to cover possible losses.

Various risk measures have been proposed: the standard deviation of the investment return distribution, the interquartile range or shortfall measures. Value at Risk (VaR) is the most popular measure in practice. It is a function of two parameters, the percentile probability \( \alpha \) and the time horizon \( T \), and is defined as the loss associated with the \( \alpha \) percentile of the portfolio return distribution over \([0, T]\).

**Definition 1.1** Let \( X \) be the loss random variable over a fixed period \([0, T]\) and \( 0 < \alpha < 1 \), the Value at Risk of \( X \) is given by:

\[
VaR_\alpha = \inf\{x \in \mathbb{R} \mid P(X \leq x) \geq \alpha\}. \tag{1}
\]

VaR is simple to compute and to interpret, as it always represents a loss, which explains its universal acceptance. Yet, several authors have pointed out the deficiencies of VaR. It is difficult to optimize; as a non-convex measure it may have multiple local minima (Basak y Shapiro, 2001). More importantly, VaR is not a sub-additive measure (Artzner et al. 1997, 1999), that is, the VaR of a composite portfolio may be larger than the sum of all sub-portfolio VaRs. This would imply that diversification may increase risk.

Artzner et al. (1997) give desirable properties that an ideal “coherent” measure should have: sub-additivity, translation invariance, positive homogeneity and monotonicity. Such risk measures have a clear advantage over VaR and have been studied extensively in the literature:

- There have been attempts to apply the notion of coherent risk measure to the capital requirements of insurance companies [Artzner (2001)].
- Relationships have been sought between coherent risk measures and various economic parameters, as the aversion to risk [Acerbi (2001)].
- Different coherent risk measures have been defined, for instance Conditional Value at Risk [Pflug (2000), Acerbi et al. (2001, 2002a,b) or Longin (2001)].
- Many authors apply coherent risk measures to portfolio optimization [Andersson et al. (2000) or Rockafeller and Uryasev (2000, 2001)].
All these studies are restricted to static risk measures, for a fixed investment period.

In many cases treating a multi-period problem with measures defined on succession of single periods can lead to erroneous conclusions. In practice portfolios often present multi-period risks, due to the presence of intermediate cash-flows. For instance, a future generates daily “market to market” cash-flows. If the position is liquidated over a period longer than a day, multi-period cash-flows appear.

Dynamic risk measures come into play when risks are aggregated over two or more periods. According to Basak and Shapiro (2001), risk management often reduces to an optimization problem because of regulatory or other risk control requirements. If the optimization problem is dynamic and risk management is required, then it is natural that the risk measure also be dynamic.

Furthermore, we believe that setting the risk measures in a dynamic framework can give additional information in taking investment decisions. If a dynamic risk measure is computed at the initial investment time, tomorrow’s risk exposure evaluation will be taken into account in today’s investment decision. By contrast, a static risk measure would not account for future exposures.

The literature on risk measures in a multi-period setting is rather limited. Cvitanić y Karatzas (1999) introduce a risk measure in a dynamic setting through a maximin criterion. Their objective is to cover a liability within a given time horizon. Basak and Shapiro (2001) use a continuous time model to maximize the utility at equilibrium, under a static VaR constraint. Ahn et al. (1999) minimize the VaR of an institution using options, also in continuous time, given the institution’s exposure to risk market.

Here we define a generalization of coherent risk measures to a dynamic setting, by proposing four properties these should meet. The dynamic coherent risk measures are characterized over a set of scenarios and possible rebalancings of the portfolio.

We focus on risk measures based on the Choquet integral. That is, measures that can be expressed as the expected value of future payments (financial assets) or future risks (insurance liabilities) with respect to a “risk neutral non-additive probability”, obtained through a probability distortion operator. VaR and Tail Conditional Expectation (TCE) are examples of such measures. Using different updating rules to incorporate market information to these measures, we define dynamic risk measures, like a dynamic VaR and dynamic TCE.
2 Dynamic coherent risk measures

Artzner et al. (1997) propose a set of properties for a risk measure to be coherent. They show that these measures are function of scenarios; in fact the choice of a risk measure corresponds to the choice of a set of generalized scenarios.

Let $\Omega$ be the set of states of nature, assumed finite with $\text{card}(\Omega) = n$. Define a $\sigma$-algebra $\mathcal{F}$, on $\Omega$. The random variable $X : \Omega \rightarrow \mathbb{R}$, represents the loss of an initial investment (risk). Consider a single uncertainty period $[0, T]$ and let $X$ be the set of all risks, that is, all $\mathcal{F}$-measurable real functions on $\Omega$.

**Definition 2.1** A risk measure is a function $\rho : X \rightarrow \mathbb{R}$.

Artzner et al. (1997) propose axioms of sub-additivity, positive homogeneity, translation invariance and monotonicity (see Definition 2.3) that risk measures must verify in order to be coherent. These coherent risk measures are defined on a single uncertainty period.

Instead, here we consider $N + 1$ dates $T = \{t_0 = 0, \ldots, t_N = T\}$, where risks are stochastic processes, depending both on time and the state of nature.

Define a filtration $\{\mathcal{F}_t\}$ on the sample space $(\Omega, \mathcal{F})$, where $\mathcal{F}_t \subseteq \mathcal{F}_{t+1}$ and $\mathcal{F}_t$ is a sub-$\sigma$-algebra of $\mathcal{F}$; at any time $t$ it represents the information available up to that instant. Now let $\mathbb{P}$ be a probability measure on $(\Omega, \mathcal{F})$.

Assume that on this (finite) set of dates $T = \{t_0 = 0, \ldots, t_N = T\}$, all possible prices of financial (assets) or insurance (liabilities) instruments form a set $I = \{I_1, I_2, \ldots, I_m\}$, which is also finite. For convenience denote gains (assets) as $I_i(w, t) < 0$, for $i = 1, \ldots, k$, while losses (liabilities) are denoted by $I_i(w, t) \geq 0$, for $i = k + 1, \ldots, m$.

For fixed $w \in \Omega$ and $t \in T$, let $X(w, t) = x = (x_1, x_2, \ldots, x_m) \in \mathbb{R}^m$ be the investment position. Each $x_i$ indicates the number of $I_i$ in this $\mathcal{F}_t$-adapted portfolio $X$. We say that such portfolios are self-financed if:

$$\sum_{i=1}^{m} X_i(w, t_{j-1}) I_i(w, t_j) = \sum_{i=1}^{m} X_i(w, t_j) I_i(w, t_j), \quad \forall w \in \Omega, \quad j = 1, \ldots, N. \quad (2)$$

Assume that the first asset is the numeraire, $I_1(w, t_j) = -1$, for all $w \in \Omega$ and $j = 1, \ldots, N$, and consider the set $\mathcal{L}(I_1, \ldots, I_m)$ of all possible payments reachable
at liquidation time $T$ through these instruments:

$$
\mathcal{L}(I_1, \ldots, I_m) = \{ Y : \Omega \to \mathbb{R} \mid \exists \ e = (e_1, \ldots, e_m) \in \mathbb{R}^m \\
\text{and } Y(w) = \sum_{i=1}^m e_i I_i(w, T), \forall w \in \Omega \}.
$$

(3)

Static risk measures are defined on this set as follows.

**Definition 2.2** A static risk measure is a decreasing function

$$
m : \mathcal{L}(I_1, \ldots, I_m) \to \mathbb{R},
$$

in the sense that $m(Y) \leq m(Y')$ for any two reachable payment variables $Y, Y' \in \mathcal{L}(I_1, \ldots, I_m)$ such that $Y(\omega) \leq Y'(\omega)$, for all $\omega \in \Omega$.

This definition is more specific than that of Artzner et al. (1997) who define risk measures on the set of all risks $X$ (see Definition 2.1). Here risk measures are defined on the set $\mathcal{L}(I_1, \ldots, I_m)$ of reachable payments, a subset of $X$, corresponding to the possible risk outcomes with an initial portfolio investment of $x$.

Now let $\mathcal{M}$ be the set of adapted stochastic processes on $(\Omega, \{F_t\})$. Each initial investment $x = (x_1, \ldots, x_m)$ defines a dynamic risk measure, as the stochastic process representing the risk associated with time $t \in T$ and outcome $w \in \Omega$.

**Definition 2.3** A dynamic risk measure is an arbitrary stochastic process

$$
M : \mathbb{R}^m \to \mathcal{M}
\quad
x \to \rho_x.
$$

**Remark 2.1** Note that at $T$, no risk remains and hence the process ends at $t_{N-1}$.

**Axiom S.** *Sub-additivity:* For $x, y \in \mathbb{R}^m$ fixed, $\rho_{x+y}(w, t) \leq \rho_y(w, t) + \rho_x(w, t)$ for all $w \in \Omega$ and $t \in \{t_0 = 0, \ldots, t_{N-1}\}$.

This property ensures that diversification reduces risk.

**Axiom PH.** *Positive homogeneity:* For $x \in \mathbb{R}^m$ fixed and any $\lambda \geq 0$, then $\rho_{\lambda x}(w, t) = \lambda \rho_x(w, t)$ for all $w \in \Omega$ and $t \in \{t_0 = 0, \ldots, t_{N-1}\}$.
The axioms of sub-additivity and positive homogeneity, jointly imply that the risk measure is convex.

**Axiom T.** *Translation invariance:* For $x \in \mathbb{R}^m$ fixed and any $\alpha = (a, 0, 0, \ldots , 0) \in \mathbb{R}^m$, $\rho_{x+\alpha}(w, t) = \rho_x(w, t) - a$, for all $w \in \Omega$ and $t \in \{t_0, \ldots , t_{N-1}\}$.

This property ensures that, at any time, an investment of $a$ in the (risk-free) numeraire, the portfolio risk decreases by an equivalent amount.

**Axiom M.** *Monotonicity:* Let $x, y \in \mathbb{R}^m$ be such that $x_i \leq y_i$, for $i = 2, \ldots , k$ but $x_i \geq y_i$ for $i = 1$ and $i = k + 1, \ldots , m$, then $\rho_x(w, t) \leq \rho_y(w, t)$ for all $w \in \Omega$ and $t \in \{t_0, 0, \ldots , t_{N-1}\}$.

**Definition 2.4** A risk measure $M : \mathbb{R}^m \rightarrow \mathcal{M}$ that satisfies axioms S, PH, T and M is called a dynamic coherent risk measure.

Note that for a single uncertainty period, i.e. in the static case, this definition coincides with the coherent risk measure of Artzner et al. (1997).

**Definition 2.5** Given an initial investment $x \in \mathbb{R}^m$ and $w \in \Omega$, the set of all reachable payments at time $t$ through self-financed portfolios is defined as:

$$\mathcal{L}_x(w, t) = \{Y \in \mathcal{L}(I_1, \ldots , I_m) | \exists X, \text{an adapted self-financed portfolio } X(w, t) = x, \sum_{i=1}^m I_i(w', T)X_i(w', T) = Y(w') , \sum_{i=1}^m I_i(w, t)X_i(w, t) = \sum_{i=1}^m I_i(w', t)X_i(w', t), \forall w' \in \Omega\} . \quad (5)$$

We can now define a dynamic coherent risk measure giving, at each instant, the minimum risk of payments reachable through all possible rebalancings, without the addition of extra capital.

**Theorem 2.1** For any $x \in \mathbb{R}^m$, define

$$M(x) = \inf_{Y \in \mathcal{L}_x(w, t)} m[Y] , \quad (6)$$

where $m$ is a static coherent risk measure in (4). Then $M$ is a dynamic coherent risk measure.
Remark 2.2  Note that (6) defines a whole family of dynamic coherent risk measures; each static measure generates a different dynamic one. We shall study the family generated by static risk measures based on a Choquet integral with respect to a distorted probability. This includes VaR (a non-coherent measure), TCE and the WT proposed by Wang (2002), the latter two being coherent.

Proof:  1. Positive homogeneity: To show that \( \inf_{Y \in \mathcal{L}_\lambda(x, t)} m[Y] = \lambda \inf_{Y^* \in \mathcal{L}_x(w, t)} m[Y^*] \), for \( \lambda \geq 0 \), is equivalent to see that \( Y \in \mathcal{L}_\lambda(w, t) \) can be expressed as \( \lambda Y^* \), where \( Y^* \in \mathcal{L}_x(w, t) \).

Let \( X^* = \frac{1}{\lambda}X \), where \( X \) is a self-financed portfolio as in (5). Then \( X \) satisfies (2); multiplying both sides of the equality by \( \frac{1}{\lambda} \) implies that \( X^* \) satisfies (2) and hence is a self-financed portfolio, also adapted since \( X \) is.

Similarly, \( X \) satisfies (5); multiplying by \( \frac{1}{\lambda} \) shows that so does \( X^* \) and hence:

\[
X^*(w, t) = \frac{1}{\lambda}X(w, t) = \frac{1}{\lambda}x\lambda = x,
\]

\[
I(w', T) \cdot X^*(w', T) = \frac{1}{\lambda}I(w', T) \cdot X(w', T) = Y^*(w'),
\]

where \( \cdot \) is the scalar product.

2. Translation invariance: To show that \( Y \in \mathcal{L}_{x+a}(w, t) \) can be expressed as \( Y^* - a \), where \( Y^* \in \mathcal{L}_x(w, t) \), consider \( X^* = X - \alpha \), with \( X \) a self-financed portfolio satisfying (2), or equivalently:

\[
X_1(w', t'_{j-1}) + \sum_{i=2}^{m} X_i(w', t'_{j-1}) I_i(w', t'_j) = X_1(w', t'_j) + \sum_{i=1}^{m} X_i(w', t'_j) I_i(w', t'_j).
\]

Subtracting \( a \) on both sides and using \( I_1(w, t) = -1 \) for all \( w \in \Omega, t \in \mathbb{T} \), gives

\[
\sum_{i=1}^{m} X_i^*(w', t'_{j-1}) I_i(w', t'_j) = \sum_{i=1}^{m} X_i^*(w', t'_j) I_i(w', t'_j).
\]

Since \( X \) satisfies (5), a similar argument as in 1., above, gives:

\[
\sum_{i=1}^{m} I_i(w, t) X_i^*(w, t) = \sum_{i=1}^{m} I_i(w', t) X_i^*(w', t), \quad \forall w' \in \Omega.
\]

The portfolio initial investment and final payments are:

\[
X^*(w, t) = X(w, t) - \alpha = x + \alpha = x,
\]

\[
I(w', T) \cdot X^*(w', T) = Y(w') + a = Y^*(w').
\]
A final application of this result to $M$ and the fact that the static risk measure $m$ is translation invariant gives:

$$M(x + a) = \inf_{Y^* - a \in \mathcal{L}(w, t)} m[Y^* - a] = \inf_{Y^* \in \mathcal{L}(w, t)} m[Y^*] - a = M(x) - a.$$ 

3. Sub-additivity: For $x_1, x_2 \in \mathbb{R}^m$ it is easily seen that $Y \in \mathcal{L}_{x_1 + x_2}(w, t)$ can be expressed as $Y_1 + Y_2$, with $Y_1 \in \mathcal{L}_{x_1}(w, t)$ and $Y_2 \in \mathcal{L}_{x_2}(w, t)$. Now, since $m$ is sub-additive:

$$M(x_1 + x_2) \leq \inf_{Y_1 + Y_2} \{ m[Y_1] + m[Y_2] \mid Y_1 \in \mathcal{L}_{x_1}(w, t), Y_2 \in \mathcal{L}_{x_2}(w, t) \}$$

$$= \inf_{Y_1 \in \mathcal{L}_{x_1}(w, t)} m[Y_1] + \inf_{Y_2 \in \mathcal{L}_{x_2}(w, t)} m[Y_2] = M(x_1) + M(x_2).$$

4. Monotonicity: Consider $x, y \in \mathbb{R}^m$ such that $x_i \leq y_i$ for $i = 2, \ldots, k$, but $x_i \geq y_i$ for $i = 1$ and $i = k + 1, \ldots, m$. We need to show that for any $Y_1 \in \mathcal{L}_x(w, t)$, there exists a $Y_2 \in \mathcal{L}_y(w, t)$ such that $Y_2(w') \geq Y_1(w')$, for all $w' \in \Omega$. Hence define

$$X^* = X - \left( \sum_{i=1}^m x_i I_i(w, t) - y_i I_i(w, t), 0, \ldots, 0 \right),$$

where $\sum_{i=1}^m x_i I_i(w, t) - y_i I_i(w, t)$ stands for the difference in prices at $t$ of the initial investments $x$ and $y$.

Note that portfolio $X^*$ is expressed as $X^* = X - (c, 0, \ldots, 0)$, where $c$ is a constant. This is identical to the representation for translation invariance, hence it suffices to see that the payment is larger than $Y_1$:

$$\sum_{i=1}^m I_i(w', T)X_i^*(w', T) = \sum_{i=1}^m I_i(w', T)X_i(w', T) - \sum_{i=1}^m x_i I_i(w, t) - y_i I_i(w, t)$$

$$\geq \sum_{i=1}^m I_i(w', T)X_i(w', T) = Y_1(w').$$

2.1 Characterization of dynamic coherent risk measures

Artzner et al. (1997) show that static coherent risk measures are function of scenarios and that this representation is one-to-one. For each coherent risk measure there exists a set of generalized scenarios, such that the measure is represented as the supremum of expected values over a finite set of probability measures.
Theorem 2.2 A risk measure $\rho$ is coherent if and only if there exists a set $\mathbb{P}$, of probability measures defined on $(\Omega, \mathcal{F})$, continuous with respect to the true probability measure, such that:

$$\rho(X) = \sup_{P \in \mathbb{P}} \{E_P[X]\} .$$

By definition, the same result holds for dynamic coherent risk measures.

Theorem 2.3 Let $M$ be a dynamic coherent risk measure defined in (6). If $m$ is static coherent risk measure, then there exists a sequence $\{\mathbb{P}_t\}_{t=0}^{T-1}$, of sets of probability measures defined on $(\Omega, \mathcal{F})$, such that:

$$M(x) = \inf_{Y \in \mathcal{L} \times (w, t)} \sup_{P \in \mathbb{P}_t} E_P[Y] .$$

Proof: apply Theorem 2.2 to the static coherent risk measure $m$. \qed

Remark 2.3 These sets of scenarios can be given explicitly in the case of risk measures based on Choquet integrals.

3 Measures based on distortions

This section focuses on risk measures defined through a Choquet integral with respect to a distorted probability. We first review the theory of non-additive measures of Dennenberg (1994) and its relation with the premium principles of Yaari (1987), Dennenberg (1990) and Wang (1996).

3.1 Distorted probabilities

Consider the set of functions $\mu : \mathcal{F} \to [0, \infty)$, defined on a $\sigma$-algebra $\mathcal{F}$, that are monotone, such that $\mu(\emptyset) = 0$ and that $A \subseteq B \Rightarrow \mu(A) \leq \mu(B)$, for any $A, B \in \mathcal{F}$. This set is called sub-modular if:

$$\mu(A \cup B) + \mu(A \cap B) \leq \mu(A) + \mu(B) \quad \text{for all } A, B \in \mathcal{F} ,$$

and $\mu$ is called sub-additive as inequality holds when $A \cap B = \emptyset$. 

An interesting special case is when \( \mu(A) = g[P(Y \in A)] \), where \( g : [0, 1] \to [0, 1] \) is an increasing function, with \( g(0) = 0 \) and \( g(1) = 1 \), \( P \) is a probability measure on \( \mathcal{F} \) and \( Y \) a random variable as in Section 2. The function \( g \) is called a distortion function and \( \mu \) a distorted probability.

Dennenberg (1990) and Wang (1996) propose an insurance risk adjustment premium principle based on the expected value with respect to a distorted probability measure, a non-additive function. This premium principle is related to the theories of Yaari (1987) and Schmeidler (1989).

Yaari (1987) postulates that to every decision corresponds an underlying probability distribution. The latter is adjusted according to the individual’s risk aversion, the decision then being taken based on the adjusted probability distribution. This idea is dual to the theory of utility, in which individuals adjust their decisions according to the utility of outcomes (von Neumann y Morgenstern, 1944).

Under the proposal of Yaari (1987) and Wang (1996) actuaries obtain risk adjusted premiums from the distribution of risk \( Y \), first distorted by \( g \), by then computing the expectation of \( Y \) with respect to the distorted probability.


As in Wang (2000), our risk measure will depend on the chosen distortion function; special cases are VaR, TCE and the WT measure of Wang (2002). First, in the case of a static risk measure, consider:

\[
m[Y] = \int Y d\mu = \int Y^+ d\mu - \int Y^- d\mu,
\]

when the integrals are finite, where \( \tilde{\mu}(A) = \mu(\Omega) - \mu(A^c) \) defines the dual or conjugate of \( \mu \), which exists as it is defined on a \( \sigma \)-algebra \( \mathcal{F} \ni A \), \( Y^+ = \max(0, Y) \) is the positive part of variable \( Y \), while \( Y^- = \max(0, -Y) \) is its negative part.

Here we define \( \mu \) as a distorted probability:

\[
\mu(A) = g \circ P(A) \quad \text{and} \quad \tilde{\mu}(A) = g[P(\Omega)] - g[P(A^c)] = 1 - g[1 - P(A)] , \quad A \in \mathcal{F}.
\]

Let \( h(u) = 1 - g(1 - u) \), for \( u \in [0, 1] \), be the dual of \( g \). It is also a distortion function, since \( g \) is one:
1. \( h(0) = 1 - g(1 - 0) = 1 - 1 = 0 \) and \( h(1) = 1 - g(1 - 1) = 1 \),

2. \( h \) is an increasing function, as \( g \) is.

### 3.2 Coherent risk measures based on distortions

Here the static risk measure in (7) becomes:

\[
m[Y] = \int_0^{\infty} g[S_Y(y)]dy - \int_0^{\infty} \{1 - g[1 - S_{Y-}(y)]\}dy
\]

\[
= \int_0^{\infty} g[S_Y^+(y)]dy - \int_0^{\infty} h[S_{Y-}(y)]dy
\]

where \( S_Y(y) = P\{Y > y\}, y \in \mathbb{R}^+ \), is the survival function of \( Y \). If the latter represents an insurance risk, that is a loss \( Y = Y^+ \geq 0 \), then:

\[
m[Y] = \int_0^{\infty} g[S_Y(y)]dy
\]

where \( g \) is a concave distortion. By contrast, \( Y = -Y^- \leq 0 \) is a financial risk (gain) and

\[
m[Y] = \int_0^{\infty} h[S_Y(y)]dy
\]

where \( h \) is convex. This provides a unified treatment, simply considering gains as negative losses. This static risk measure is coherent for a concave distortion function \( g \).

**Theorem 3.1** Let \( Y \in \mathcal{L}(I_1, \ldots, I_m) \) be the random payments generated by a portfolio composed of insurance and financial risks. The static risk measure in (8) is coherent for \( g : [0, 1] \to [0, 1] \) increasing and concave, with \( g(0) = 0, g(1) = 1 \) and \( h(u) = 1 - g(1 - u) \), over \( [0, 1] \).

**Proof:** 1. **Translation invariance:**

\[
m[Y + a] = \int_0^{\infty} g[S_{Y+a}^+(y)]dy - \int_0^{\infty} h[S_{Y+a-}(y)]dy
\]

\[
= \int_0^{\infty} g[S_Y^+(y)]dy - \int_0^{\infty} h[S_{Y-}(y)]dy + a = m[Y] + a
\]
2. **Positive homogeneity:** Let \( \lambda > 0 \) (the \( \lambda = 0 \) case is trivial). By definition \( (\lambda Y)^+ = \lambda Y^+ \) and \( (\lambda Y)^- = \lambda Y^- \), which imply:

\[
m[\lambda Y] = \int_0^\infty g[S_{\lambda Y^+}(y)]dy - \int_0^\infty h[S_{\lambda Y^-}(y)]dy
= \int_0^\infty g[S_{Y^+}(s)]ds - \int_0^\infty h[S_{Y^-}(s)]\lambda ds = \lambda m[Y].
\]

3. **Monotonicity:** Let \( Y_1, Y_2 \in \mathcal{L}(I_1, \ldots, I_m) \), be such that \( Y_1(w) \leq Y_2(w) \), for all \( w \in \Omega \). Then \( Y_1^+ \leq Y_2^+ \) and \( Y_1^- \geq Y_2^- \), and in turn, \( g, h \) increasing implies:

\[
m[Y_1] = \int_0^\infty g[S_{Y_1^+}(y)]dy - \int_0^\infty h[S_{Y_1^-}(y)]dy
\leq \int_0^\infty g[S_{Y_2^+}(y)]dy - \int_0^\infty h[S_{Y_2^-}(y)]dy = m[Y_2].
\]

4. **Sub-additivity:** Since \((Y_1 + Y_2)^+ \leq Y_1^+ + Y_2^+ \) and \((Y_1 + Y_2)^- \geq Y_1^- + Y_2^- \), then

\[
m[Y_1 + Y_2] = \int_0^\infty g[S_{Y_1+Y_2^+}(y)]dy - \int_0^\infty h[S_{Y_1+Y_2^-}(y)]dy
\leq \int_0^\infty g[S_{Y_1^+}+Y_2^+](y)dy - \int_0^\infty h[S_{Y_1^-}+Y_2^-](y)dy.
\]

Dennenberg (1994) shows that the Choquet integral is sub-additive if \( \mu \) is (here when \( g \) is concave, and hence so is \( -h \)). Then

\[
m[Y_1 + Y_2] \leq \int_0^\infty g[S_{Y_1^+}(y)]dy - \int_0^\infty h[S_{Y_1^-}(y)]dy
+ \int_0^\infty g[S_{Y_2^+}(y)]dy - \int_0^\infty h[S_{Y_2^-}(y)]dy = m[Y_1] + m[Y_2].
\]


Market information can be added to improve this static risk measure. Combining the initial probability distribution to market data gives an a-posteriori probability that, according to Yaari’s theory, can be distorted as above to obtain an updated risk measure.

Classical Bayesian theory conditions the distribution of \( Y \) on the available information to obtain a predictive distribution of \( Y' \), given the observed data. In addition, applying the premium principle of Wang (1996) gives the risk-adjusted
premium as the expectation of \( Y \) with respect to this predictive distribution, hence updating the risk measure.

Here we follow Gilboa and Schmeidler (1993) and instead update the distorted probability measure with the market information. This yields a risk measure given at each instant by the expectation of future payments with respect to this updated distorted probability.

For non-additive measures, Dennenberg (1994) defines three information updating rules: Bayes’, Dempster-Shafer’s and a general rule. Using Dennenberg’s terminology and notation, consider a monotone function \( \mu \) defined on a \( \sigma \)-algebra \( \mathcal{F} \):

**Definition 3.1** The general rule to condition \( \mu \) on \( B \in \mathcal{F} \) is:

\[
\mu_B(A) = \frac{\mu(A \cap B)}{\mu(B \cap A) + \mu(A^c \cap B)}, \quad \forall A \in \mathcal{F}.
\]

If \( B \) is such that \( \mu(A \cap B) = 0 \), then \( \mu_B(A) = 0 \) by convention.

Since for the dual \( \tilde{\mu}(C) = 1 - \mu(C^c) \), for any \( C \in \mathcal{F} \), the conditional measure above can be expressed in terms of \( \mu \) alone:

\[
\mu_B(A) = \frac{\mu(B \cap A)}{\mu(B \cap A) + 1 - \mu(A \cup B^c)}, \quad \forall A \in \mathcal{F}.
\]  

(9)

**Definition 3.2** Bayes’ conditional rule for \( \mu \) given \( B \in \mathcal{F} \) is:

\[
\mu\setminus_B(A) = \frac{\mu(A \cap B)}{\mu(B)}, \quad \forall A \in \mathcal{F}.
\]

(10)

Here if \( B \) is such that \( \mu(B) = 0 \), then \( \mu\setminus_B(A) = 0 \) by convention.

**Definition 3.3** Dempster-Shafer’s conditional rule for \( \mu \) given \( B \in \mathcal{F} \) is:

\[
\mu/B(A) = \frac{\mu(A \cap B) - \mu(B^c)}{1 - \mu(B^c)}, \quad \forall A \in \mathcal{F}.
\]

(11)

Again, if \( B \) is such that \( \mu(B) = 0 \), then \( \mu/B(A) = 0 \) by convention.

All three rules reduce to Bayes’ rule when \( \mu \) is additive. In particular when \( \mu = g \circ P \) and \( g \) is the identity on \([0,1]\):

\[
\mu_B(A) = \mu\setminus_B(A) = \mu/B(A) = \frac{P(A \cap B)}{P(B)} = P(A|B),
\]

13
which is the conditional probability of $A$ given en $B$. But for a general distortion function $g$, three distinct risk measures are obtained.

1. The general conditional risk measure:

$$m_1[Y] = \int Y^+ d\mu_{(B)} - \int Y^- d\tilde{\mu}_{(B)}$$

$$= \int_0^\infty \frac{g[P\{Y^> y\} \cap B]}{1 - g[P\{Y^> y\} \cup B^c]} dy + \int_0^\infty \frac{h[P\{Y^- y\} \cap B]}{1 - h[P\{Y^- y\} \cup B^c]} dy.$$

2. Bayes’ conditional risk measure:

$$m_2[Y] = \int Y^+ d\mu_{\setminus(B)} - \int Y^- d\tilde{\mu}_{\setminus(B)}$$

$$= \int_0^\infty \frac{g[P\{Y^> y\} \cap B]}{g[P(B)]} dy + \int_0^\infty \frac{h[P\{Y^- y\} \cap B]}{h[P(B)]} dy.$$

3. Dempster-Shafer’s conditional risk measure:

$$m_3[Y] = \int Y^+ d\mu_{/(B)} - \int Y^- d\tilde{\mu}_{/(B)}$$

$$= \int_0^\infty \frac{g[P\{Y^> y\} \cap B] - g[P(B^c)]}{1 - g[P(B^c)]} dy + \int_0^\infty \frac{h[P\{Y^- y\} \cap B] - h[P(B^c)]}{1 - h[P(B^c)]} dy.$$

Incorporating market information to the distorted probability, as described above, produces coherent risk measures.

**Theorem 3.2** Let $\mu = g \circ P$, where the distortion function $g$ and the random payment $Y$ satisfy the conditions of Theorem 3.1. Then the above conditional risk measures $m_i$, $i = 1, 2, 3$, are coherent.

**Proof:** We detail the proof for $m_1$, the arguments are similar for Bayes or Dempster-Shafer’s conditional risk measures and follow the lines of the proof of Theorem 3.1.
1. **Positive homogeneity:**

\[
\begin{align*}
m_1[\lambda Y] &= \int_0^{\infty} \frac{g[P\{((\lambda Y)^+ > y) \cap B]\}}{1 - g[P\{((\lambda Y)^+ > y) \cup B^c]\} + g[P\{((\lambda Y)^+ > y) \cap B]\}] \, dy \\
&\quad + \int_0^{\infty} \frac{h[P\{((\lambda Y)^- > y) \cap B]\}}{1 - h[P\{((\lambda Y)^- > y) \cup B^c]\} + h[P\{((\lambda Y)^- > y) \cap B]\}] \, dy .
\end{align*}
\]

Let \( \lambda > 0 \) (the \( \lambda = 0 \) case is trivial). Since \((\lambda Y)^+ = \lambda Y^+ \) and \((\lambda Y)^- = \lambda Y^- \):

\[
m_1[\lambda Y] = \int_0^{\infty} \frac{g[P\{(\lambda Y^+ > y) \cap B]\}}{1 - g[P\{(\lambda Y^+ > y) \cup B^c]\} + g[P\{(\lambda Y^+ > y) \cap B]\}] \, dy \\
\quad + \int_0^{\infty} \frac{h[P\{(\lambda Y^- > y) \cap B]\}}{1 - h[P\{(\lambda Y^- > y) \cup B^c]\} + h[P\{(\lambda Y^- > y) \cap B]\}] \, dy \\
= \int_0^{\infty} \frac{g[P\{Y^+ > s \cap B]\}}{1 - g[P\{Y^+ > s \cup B^c]\} + g[P\{Y^+ > s \cap B]\}] \lambda \, ds \\
\quad + \int_0^{\infty} \frac{h[P\{Y^- > s \cap B]\}}{1 - h[P\{Y^- > s \cup B^c]\} + h[P\{Y^- > s \cap B]\}] \lambda \, ds = \lambda m_1[Y].
\]

2. **Monotonicity:** Again \( Y_1, Y_2 \in \mathcal{L}(I_1, \ldots, I_m), \) such that \( Y_1(w) \leq Y_2(w) \), for all \( w \in \Omega \) imply that \( Y_1^+ \leq Y_2^+ \) and \( Y_1^- \geq Y_2^- \). Now \( g, h \) increasing and the fact that \( P(C \cap B) \leq P(C \cap B^c) \), for any \( B, C \in \mathcal{F} \) imply:

\[
1 + g[P\{Y_1^+ > y \cap B\}] - g[P\{Y_1^+ > y \cup B^c\}] \leq 1 + g[P\{Y_2^+ > y \cap B\}] - g[P\{Y_2^+ > y \cup B^c\}],
\]

as well as a similar inequality for \( h \). The result \( m_1(Y_1) \leq m_2(Y_2) \) follows.

3. **Sub-additivity:** From Dennenberg (1994) we used that \( \mu \) sub-additive implies \( \int [Y_1 + Y_2] \, d\mu \leq \int Y_1 \, d\mu + \int Y_2 \, d\mu \). He also shows the inverse inequality if \( \mu \) is super-additive. Here \( g \) concave implies \( \mu \) sub-additive and \( \tilde{\mu} \) super-additive.

Furthermore, Dennenberg (1994) shows that \( \mu \) sub-additive \( \Rightarrow \mu|_B \) sub-additive, while \( \tilde{\mu} \) super-additive \( \Rightarrow \tilde{\mu}|_B \) super-additive. Then the result \( m_1[Y_1 + Y_2] \leq m_1[Y_1] + m_1[Y_2] \) follows.

4. **Translation invariance:** As in the proof of Theorem 3.1, for \( a \in \mathbb{R} \) we divide \( Y \in \mathcal{L}(I_1, \ldots, I_m) \) in two. Considering \( S_{(Y+a)^+}(y) \) and \( S_{(Y+a)^-}(y) \) separately, the result \( m_1[Y] = m_1[Y] + a \) follows. \( \square \)
The three static conditional risk measures, above, can be updated with market information at each instant \( t \). In turn, they generate dynamic conditional risk measures that can be used with mixed portfolios, composed of both financial and insurance risks:

\[
M_i(x) = \inf_{Y \in \mathcal{L}_x(w,t)} m_i[Y], \quad \text{for } i = 1, 2, 3.
\]

In terms of distorted probabilities, these become:

\[
M_1(x) = \inf_{Y \in \mathcal{L}_x(w,t)} \int_0^\infty \frac{g[P\{Y^+ > y\} \cap B]}{1 - g[P\{Y^+ > y\} \cup B^c]} \, dy - \int_0^\infty \frac{h[P\{(Y^- > y) \cap B\}]}{1 - h[P\{(Y^- > y) \cup B^c\}]} \, dy,
\]

\[
M_2(x) = \inf_{Y \in \mathcal{L}_x(w,t)} \int_0^\infty \frac{g[P\{Y^+ > y\} \cap B]}{g[P(B)]} \, dy - \int_0^\infty \frac{h[P\{(Y^- > y) \cap B\}]}{h[P(B)]} \, dy,
\]

\[
M_3(x) = \inf_{Y \in \mathcal{L}_x(w,t)} \int_0^\infty \frac{g[P\{Y^+ > y\} \cap B]}{1 - g[P(B^c)]} \, dy - \int_0^\infty \frac{h[P\{(Y^- > y) \cap B\}]}{1 - h[P(B^c)]} \, dy,
\]

where the set \( B \in \mathcal{F}_t \) stands for the market information available at time \( t \).

Since static conditional risk measures were seen to be coherent in Theorem 3.2, their dynamic counterparts \( M_i \), above, must also be coherent (see Theorem 2.1).

### 3.3 Characterization of risk measures based on distorted probabilities

The previous section shows how the risk measure \( M \) is function of scenarios. More specifically, there exists a series of sets of scenarios, such that the risk measure is expressed as the supremum of a series de expectations:

\[
M(x) = \inf_{Y \in \mathcal{L}_x(w,t)} \sup_{P \in \mathcal{P}_t} E_P[Y].
\]

When risk measures are based on the Choquet integral, these sets of scenarios are known explicitly. Dennenberg (1994) shows how non-additive measures can be expressed as the supremum of a set of probability measures:

\[
\mu(A) = \max_{\alpha \in \text{core}(\mu)} \alpha(A),
\]
where \( \text{core}(\mu) = \{ \alpha \mid \alpha \text{ is additive on } \mathcal{F}, \text{ with } \tilde{\mu} \leq \alpha \leq \mu \} \).

Hence the dynamic risk measure \( M \) can be written in terms of a set \( \text{core}(\mu) \), of probability measures:

\[
M(x) = \inf_{Y \in \mathcal{L}_s(w, t)} \sup_{\alpha \in \text{core}(\mu)} m[Y],
\]

where \( m[Y] = \int_0^\infty g[S_{Y+}(y)]dy - \int_0^\infty h[S_{Y-}(y)]dy \). Note that \( \text{core}(\mu) \) does not depend on time \( t \).

In particular, the non-additive measures \( \mu \) can be chosen as one of the three conditional risk measures, considered above, that update the distorted probability with market information. In this case the set \( \text{core}(\mu) \) is also known explicitly. Dennenberg (1994) shows that for the general rule:

\[
\mu_{|B}(A) = \max_{\alpha \in \text{core}(\mu)} \alpha_{|B}(A), \quad \tilde{\mu}_{|B}(A) = \min_{\alpha \in \text{core}(\mu)} \alpha_{|B}(A),
\]

which yields a dynamic risk measure \( M_1(x) \) given by:

\[
M_1(x) = \inf_{Y \in \mathcal{L}_s(w, t)} \int Y^+ d\mu_B - \int Y^- d\tilde{\mu}_B = \inf_{Y \in \mathcal{L}_s(w, t)} \sup_{\alpha \in \text{core}(\mu)} \int Y d\alpha_{|B},
\]

where \( B \in \mathcal{F}_t \) and \( w \in B \).

A similar definition can be given for Bayes’ rule:

\[
\mu_{\wedge B}(A) = \max_{\alpha \in \mathcal{M}_B} \alpha_{\wedge B}(A), \quad \tilde{\mu}_{\wedge B}(A) = \min_{\alpha \in \mathcal{M}_B} \alpha_{\wedge B}(A),
\]

where \( \mathcal{M}_B = \{ \alpha \mid \alpha \in \text{core}(\mu), \text{ with } \alpha(B) = \mu(B) \} \). The corresponding dynamic risk measure \( M_2(x) \) is:

\[
M_2(x) = \inf_{Y \in \mathcal{L}_s(w, t)} \int Y^+ d\mu_{\wedge B} - \int Y^- d\tilde{\mu}_{\wedge B} = \inf_{Y \in \mathcal{L}_s(w, t)} \sup_{\alpha \in \mathcal{M}_B} \int Y d\alpha_{\wedge B}.
\]

This set \( \mathcal{M}_B \) of scenarios is also obtained for Dempster-Shafer’s rule, giving:

\[
M_3(x) = \inf_{Y \in \mathcal{L}_s(w, t)} \int Y^+ d\mu_{/B} - \int Y^- d\tilde{\mu}_{/B} = \inf_{Y \in \mathcal{L}_s(w, t)} \sup_{\alpha \in \mathcal{M}_B} \int Y d\alpha_{/B}.
\]

Note how these sets of scenarios now depend on the time variable \( t \).
4 Examples

Using the above results, we embed three popular static risk measures in a dynamic framework. All can be expressed as Choquet integrals with respect to a distorted probability. These are VaR, a non-coherent static measure, plus two coherent static risk measures: TCE and the measure WT, proposed by Wang (2002).

4.1 Value at Risk

Value at Risk, see (1), is a risk measure that only uses information on the frequency of losses, not their severity. For instance, doubling the severity of the largest loss would not affect VaR. Clearly VaR is a popular and useful risk measure in practice, but it also suffers important drawbacks (see Wirch, 1999).

VaR can be expressed as a Choquet integral with respect to a distorted probability measure, using the following distortion function:

\[ g(y) = \begin{cases} 
0 & \text{si } y < 1 - \alpha \\
1 & \text{si } y \geq 1 - \alpha 
\end{cases} \]

This function is non-decreasing, piece-wise defined with \( g(0) = 0 \) and \( g(1) = 1 \), but not concave. Hence its associated risk measure is not coherent (Wirch and Hardy, 1999):

\[
VaR_{\alpha} = \int_{y_{\alpha}}^{\infty} g[S_{Y^+}(y)]dy - \int_{0}^{\infty} h[S_{Y^-}(y)]dy = \int_{0}^{V_{\alpha}} dy = V_{\alpha},
\]

where \( V_{\alpha} \) is the corresponding percentile of the distribution of \( Y \).

A dynamic VaR measure is obtained by conditioning the distorted probability measure \( \mu = g \circ P \) on the market information available at \( t \), say \( B \in \mathcal{F}_t \). This can be done using any of the three rules on the previous section, each defining a different dynamic risk measure.

Using Bayes’ rule (the process is similar for the other two rules) gives:

\[
m_2[Y] = \int_{0}^{\infty} \frac{g[P\{(Y^+ > y) \cap B\}]}{g[P(B)]} dy + \int_{0}^{\infty} \frac{h[P\{(Y^- > y) \cap B\}]}{h[P(B)]} dy,
\]
where the distortion functions $g$ and $h$ are given by:
\[
g[P\{Y^+ > y\} \cap B] = \begin{cases} 
0 & \text{if } 0 \leq P\{Y^+ > y\} \cap B \leq 1 - \alpha \\
1 & \text{if } 1 - \alpha \leq P\{Y^+ > y\} \cap B \leq 1 
\end{cases},
\]
\[
h[P\{Y^- > y\} \cap B] = \begin{cases} 
0 & \text{if } 0 \leq P\{Y^- > y\} \cap B \leq 1 - \alpha \\
1 & \text{if } 1 - \alpha \leq P\{Y^- > y\} \cap B \leq 1 
\end{cases}.
\]

For a given distribution of $Y$, single out the different possible events in the above probability statements. Since $P\{Y > y\} \in B \in \mathcal{F}_t$, then:

1. If $0 \leq P\{Y > y\} \cap B \leq 1 - \alpha$ and $1 - \alpha \leq P(B) \leq 1$, then $m_2[Y] = 0$ and the risk at that instant has measure zero.

2. If $1 - \alpha \leq P\{Y > y\} \cap B \leq 1$, then $1 - \alpha \leq P(B) \leq 1$, implying that $g[P\{Y > y\} \cap B] = g[P(B)] = 1$ and hence $m_2[Y] = V^*_\alpha$, where $V^*_\alpha$ is the corresponding percentile of the conditional distribution of $Y$, given $B \in \mathcal{F}_t$.

3. If $0 \leq P(B) \leq 1 - \alpha$ then $0 \leq P\{Y > y\} \cap B \leq 1 - \alpha$, in which case we can assume that the risk measure is zero at that instant.

Putting together all possible events gives the dynamic risk measure:
\[
M_2(x) = \begin{cases} 
\inf_{Y \in \mathcal{L}_x(w,t)} V^*_\alpha & \text{if } 1 - \alpha \leq P(B) \leq 1 \\
0 & \text{if } 0 \leq P(B) \leq 1 - \alpha
\end{cases},
\]
i.e. for sufficiently large information sets $B \in \mathcal{F}_t$ (with probability larger than $\alpha$), the risk at that instant $t$ corresponds to the minimum conditional VaR of all payments $Y$, reachable through rebalancings of a portfolio with initial investment $x$, given the available market information $B$.

### 4.2 Tail Conditional Expectation

As a coherent alternative to VaR, Artzner et al. (1999) propose the Tail Conditional Expectation (TCE) risk measure, also called Tail-VaR. In fact, when the loss distribution is continuous, TCE defines a measure equivalent to Conditional Value at Risk (CVaR) and Expected Shortfall (ES). But this equivalence does not hold in general (Acerbi, 2001).
TCE is defined as the conditional expected loss, given that it is larger than the \( \alpha \)-percentile of the loss distribution, that is (when the latter is continuous):

\[
TCE(\alpha) = E[Y \mid Y > VaR_{\alpha}] .
\]  

(12)

In general (see Hardy, 2001):

\[
TCE(\alpha) = VaR_{\alpha} + \frac{P\{Y > VaR_{\alpha}\}}{(1 - \alpha)} E[Y - VaR_{\alpha} \mid Y > VaR_{\alpha}] ,
\]

since for general distributions it may be that \( P\{Y > VaR_{\alpha}\} < 1 - \alpha \).

Unlike VaR, TCE does not only reflect the loss frequency but also the expected in excess of the VaR. It is coherent and in general a better risk measure than VaR. In Canada, the Office of the Superintendent of Financial Institutions uses a TCE(0.95) to determine capital requirements for insurance companies. Yet, TCE only reflects losses in excess of the VaR. It does not fit properly to extreme losses that occur with a low frequency, as it measures the mean excess-losses.

A Choquet integral representation can also be to TCE, using the following distortion function:

\[
g(y) = \begin{cases} 
\frac{y}{1-\alpha} & \text{if } y \leq 1 - \alpha \\
1 & \text{if } y \geq 1 - \alpha
\end{cases}
\]

where \( g \) is non-decreasing, continuous with \( g(0) = 0 \), \( g(1) = 1 \) and concave (hence coherent). As for VaR, we can define a conditional risk measure based on one of the updating rules. Using Bayes’ rule the distorted probability measure becomes:

\[
g[P\{Y > y \cap B\}] = \begin{cases} 
\frac{P\{Y > y \mid \cap B\}}{1 - \alpha} & \text{if } 0 \leq P\{Y > y \cap B\} \leq 1 - \alpha \\
1 & \text{if } 1 - \alpha \leq P\{Y > y \cap B\} \leq 1
\end{cases}
\]

where, as for VaR, three cases can be distinguished:

1. If \( 0 \leq P(B) \leq 1 - \alpha \) then \( 0 \leq P\{Y > y \cap B\} \leq 1 - \alpha \) and hence

\[
g[P\{Y > y \cap B\}] = \frac{P\{Y > y \cap B\}}{1 - \alpha} \quad \text{and} \quad g[P(B)] = \frac{P(B)}{1 - \alpha} ,
\]

implying that

\[
\frac{g[P\{Y > y \cap B\}]}{g[P(B)]} = \frac{P\{Y > y \cap B\}}{P(B)} .
\]
2. If $1 - \alpha \leq P\{(Y > y) \cap B\} \leq 1$, then as for VaR, $1 - \alpha \leq P(B) \leq 1$ and $g\left[P\{(Y > y) \cap B\}\right] = g[P(B)] = 1$, which here gives

$$\frac{P\{(Y > y) \cap B\}}{P(B)} = 1.$$ 

3. Finally, if $0 \leq P(B) \leq 1 - \alpha$ and also $1 - \alpha \leq P\{(Y > y) \cap B\} \leq 1$, then

$$\frac{g\left[P\{(Y > y) \cap B\}\right]}{g[P(B)]} = \frac{P\{(Y > y) \cap B\}}{1 - \alpha}.$$ 

Again, the risk measure depends $P(B)$. For small information sets with probability smaller than $1 - \alpha$, the static risk measure is proportional to the conditional expected loss, given the available information:

$$m_2[Y] = k(\alpha, \beta)E[Y \mid B], \quad \text{if } 0 \leq \beta \leq 1 - \alpha,$$

where $\beta = P(B)$ and the constant $k(\alpha, \beta) = \frac{\beta}{1 - \alpha} \leq 1$. The corresponding dynamic risk measure is then

$$M_2(x) = k[\alpha, P(B)] \inf_{Y \in \mathcal{L}_{x}(w,t)} E[Y \mid B], \quad \text{if } 0 \leq P(B) \leq 1 - \alpha.$$ 

By contrast, for sufficiently large information sets $B \in \mathcal{F}_t$, with probability $1 - \alpha \leq P(B) \leq 1$, then:

$$\frac{g\left[P\{(Y > y) \cap B\}\right]}{g[P(B)]} = \begin{cases} 1 & \text{if } 1 - \alpha \leq P\{(Y > y) \cap B\} \leq 1 \\ \frac{P\{(Y > y) \cap B\}}{P(B)} & \text{if } 0 \leq P\{(Y > y) \cap B\} \leq 1 - \alpha \end{cases},$$

giving a static risk measure $m_2$, proportional to the $TCE^*(\alpha)$ of the conditional losses, given the information available at that instant. Again as for VaR, the proportionality holds for the dynamic TCE risk measure:

$$M_2(x) \propto \inf_{Y \in \mathcal{L}_{x}(w,t)} TCE^*(\alpha), \quad \text{if } 1 - \alpha \leq P(B) \leq 1.$$ 

### 4.3 The WT risk measure

The previous section shows how the TCE risk measure, although coherent, only accounts for losses in excess of the VaR. This is clearly observed in the truncation of the distortion function that characterizes TCE.
Wang (2002) gives various illustrations of coherent measures based on “the whole probability distribution”. He proposes a smooth (differentiable) distortion called Wang’s transform:

\[ g_\lambda(x) = \Phi[\Phi^{-1}(x) - \lambda], \quad x \in (0, 1), \]

where \( \Phi \) is the standard normal distribution function and \( \lambda \in \mathbb{R} \).

Wang (2000) defined the WT risk measure, based on \( g_\lambda \), which unifies financial and insurance pricing theories. For financial risks it reproduces the Capital Asset Pricing Model (CAPM) and Black-Scholes option pricing formula, under a normal returns assumption. In addition, Wang (2001) shows that the WT transform derives from the equilibrium pricing model of Bühlmann (1980).

**Definition 4.1** Let \( Y \) be a risk with distribution \( F \). For a given confidence level \( \alpha \), set \( \lambda = \Phi^{-1}(\alpha) \) and apply Wang’s distortion to the loss distribution

\[ F^*(x) = \Phi\{\Phi^{-1}[F(x)] - \lambda\}, \quad x \in \mathbb{R}. \]

Then the WT risk measure is defined as \( WT(\alpha) = E^*[Y] \), where the expectation is under the distorted distribution \( F^* \).

For normal risks the WT measure reproduces VaR. Wang (2002) also compares it to TCE. A dynamic WT risk measure is easily be defined, first using one of the three information updating rules of Section 3.2, and then obtaining the minimum over all possible portfolio rebalancings.

**Conclusion**

Recently various authors have proposed coherent measures of risk, all in a static framework (single cash-flow period). Here we present a dynamic setting with a finitely different transaction dates and finite sample space.

For a given initial investment position, we define dynamic risk measures as stochastic processes, whose values represent the risk taken at each instant and state of nature. The properties imposed on these dynamic risk measure are: translation invariance, positive homogeneity, monotonicity and sub-additivity. These yield the properties of Artzner et al. (1997) in the special case of static measures.
By construction, each static risk measure defines a dynamic one, as the minimum risk over payments reachable through all possible portfolio rebalancings. These meet the above properties, producing coherent dynamic risk measures.

In particular, we consider measures with a Choquet integral representation, with respect to a distorted probability measure. Using three updating rules to account for market information, we propose three conditional dynamic measures that are simple to compute and generalize VaR and TCE to a dynamic setting.

References


