Unifying discrete structural credit risk models and reduced-form models

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July 15, 2002

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SUMMARY

In a structural credit risk model, a default event is triggered by the capital structure when the value of the obligor falls below its financial obligation. In a reduced-form model, the bond price of a firm is considered as the current value under risk neutral valuation of a contingent claim paying full obligation (of one unit, say) if no default event happens and paying a fraction of promised payment or nothing if a default event happens. With model-specific assumptions, the bond price can be reduced to the mean of recovery rate and the probability of default for discrete models or an intensity process for continuous models. Based on a no-arbitrage assumption, we show that the yield-spread formula for a reduced-form model is equivalent to a credit-spread formula for a structural model if the value of the firm follows a diffusion process or a jump-diffusion process. If the strong priority rule is applied, the prices of bonds of different seniority classes reflect the distribution function for the jump size and jump frequency. Also, we shows that the forward credit spread can be given by $f^i(t,T) - f(t,T) = \tilde{\mu}(T)\left\{1 - \mathrm{E}_t^Q[\delta(\tau)]\frac{P(t,T)}{V(t,T)}\right\}$.

Key words and phrases: Default risk; Brownian motion; Jump-diffusion process; Structural model; Reduced-form model.

1 Merton's structural model define a reduced-form model

1.1 Merton's structural model

Merton pioneered the structural credit risk model in 1974. He assumes that the market is perfect and frictionless thus there are many investors who can sell, short and buy as much as they want in continuous time. The term structure for interest rate is assumed to be flat and the interest is fixed for all time for both borrowing or lending. The Modigliani-Miller theorem holds so that the value of the firm is invariant to its capital structure. The value of a firm V_t at time t is assumed to follow a Brownian motion as

$$dV_t = (\alpha V_t - C)dt + \sigma V_t dZ_t \tag{1}$$

where σ^2 is the instantaneous variance, α is instantaneous return, C is total payout including dividends or coupons, and Z_t is a standard Brownian motion. A default event can only occur at its bond's maturity and occurs only if the value of the firm V_T at bond maturity T falls below the bond obligation B.

A derivative of the value of the firm $C(V_t, t)$ must follow another geometric Brownian motion as

$$dC(V_t, t) = [\alpha_c C(V_t, t) - C_c]dt + \sigma_c C(V_t, t)dZ_c. \tag{2}$$

Under the argument of no risk and no arbitrage of a representative agent's trading strategy, Merton (1974) derives the differential equation

$$\frac{1}{2}\sigma^2 V_t^2 \frac{\partial^2 C(V_t, t)}{\partial V_t^2} + r\frac{\partial C(V_t, t)}{\partial V_t} + \frac{\partial C(V_t, t)}{\partial t} - rC(V_t, t) = 0 \tag{3}$$

with boundary condition $C(x, T) = g(x)$ for a payoff $g(x)$ of a derivative at maturity time T. Let $R(t, T)$ represent the yield to maturity at time t of a risky debt that matures at time T. With the boundary condition, the credit spread is derived by Merton as

$$R(t, T) - r = -\frac{1}{T - t}\log\left\{\Phi[h_2(t)] + \frac{1}{d}\Phi[h_1(t)]\right\} \tag{4}$$

where

$$\begin{aligned} d(t) &= \frac{B\exp[-r(T-t)]}{V_t}, \\ h_1(t) &= -\frac{\ln(\frac{V_t}{B}) + (r + 1/2\sigma^2)(T-t)}{\sigma(T-t)^{1/2}} = -x_1(t), \\ h_2(t) &= \frac{\ln(\frac{V_t}{B}) + (r - 1/2\sigma^2)(T-t)}{\sigma(T-t)^{1/2}} = x_2(t), \end{aligned} \tag{5}$$

$R(t, T)$ is the yield to maturity for the bond maturing at time T, and r is the riskless rate. The value of the ratio $\frac{V_T}{B}$ determines whether there is default or not. If $\frac{V_T}{B} \geq 1$, there is no default. If $\frac{V_T}{B} < 1$, then default happens. When a default event happens, the ratio $\frac{V_T}{B}$ represents the recovery rate.

1.2 Default probability and mean recovery rate in the Merton's model

From Itô's lemma, the value of a firm in the Merton (1974) structural model follows

$$d\ln V_t = \left(\alpha - \frac{C}{V_t} - \frac{\sigma^2}{2}\right) dt + \sigma dZ_t. \tag{6}$$

If the drift $\left(\alpha - \frac{C}{V_t} - \frac{\sigma^2}{2}\right)$ does not equal to $\left(r - \frac{\sigma^2}{2}\right)$, then the representative agent has an arbitrage opportunity. Thus, the value of the firm should follow

$$d\ln V_t = \left(r - \frac{\sigma^2}{2}\right) dt + \sigma dZ_t. \tag{7}$$

At time T, $\ln V_T$ will follow a normal distribution

$$\ln V_T \sim N\left(\ln V_0 + \left(r - \frac{\sigma^2}{2}\right)T, \sigma\sqrt{T}\right). \tag{8}$$

At time t, the probability that the firm will default at time T is

$$\begin{aligned}
\Pr\{\tau = T\} &= \Pr\{V_T < B\} \\
&= \Pr\left\{\frac{\ln V_T - \ln V_t - (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} < \frac{\ln B - \ln V_t - (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right\} \\
&= \Phi\left(-\frac{\ln\frac{V_t}{B} + (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right) \\
&= \Phi(-x_2(t)).
\end{aligned} \tag{9}$$

At time t, the probability that the firm will not default at time T is

$$\begin{aligned}
\Pr(\tau > T) &= 1 - \Phi(-x_2(t)) \\
&= \Phi(x_2(t)).
\end{aligned} \tag{10}$$

The probability that $\tau < T$ is zero since default before maturity is not allowed in Merton's model.

The value $\frac{V_T}{B}$ determines whether a default event happens or not at time T. The recovery rate $\delta(T)$ is the proportion V_T of B if the firm defaults at time T. Without considering the seniority class of the bond, the recovery rate can be expressed as

$$\delta(T) = \frac{V_T}{B}\,|_{V_T<B} \tag{11}$$

$$= \frac{B - (B - V_T)_+}{B}\,|_{V_T<B} \tag{12}$$

where $\frac{B-(B-V_T)_+}{B}$ is the proportion the bond holder receives at the maturity time T. The expected value at time t of the bond holder's position is

(13)

$$
\begin{aligned}
E_t^Q\left[\frac{B-(B-V_T)_+}{B}\right] &= \frac{B-\mathrm{E}_t^Q[B-V_T]_+}{B} \\
&= \frac{B-p_te^{rT}}{B} \qquad (14)\\
&= \frac{B-e^{rT}[c_t+Be^{-rT}-V_t]}{B} \qquad (15)\\
&= \frac{V_te^{rT}-c_te^{rT}}{B} \\
&= \frac{V_te^{rT}-[V_t\Phi(x_1(t))-Be^{-rT}\Phi(x_2(t))]e^{rT}}{B} \qquad (16)\\
&= \frac{V_te^{rT}[1-\Phi(x_1(t))]+B\Phi(x_2(t))}{B} \qquad (17)\\
&= \frac{V_te^{rT}}{B}\Phi(-x_1(t))+\Phi(x_2(t)) \qquad (18)\\
&= \frac{\Phi(h_1(t))}{d}+\Phi(h_2(t)) \qquad (19)
\end{aligned}
$$

where p_t in equation (14) represents the time-t price an European put option. In equation (15), we use put-call parity and c_t represents the time-t price of an European call option. In equation (16) we substitutes c_t with the Black-Scholes-Merton call option price.

The mean recovery rate can be expressed as a function of the position of the bond holder as

$$
\begin{aligned}
\mathrm{E}_t^Q[\delta(T)] &= \frac{\mathrm{E}_t^Q\left[\frac{B-(B-V_T)_+}{B}\right]-1\,\Pr(V_T\geq B)}{\Pr(V_T<B)} \qquad (20)\\
&= \frac{\frac{\Phi(h_1(t))}{d(t)}+\Phi(h_2(t))-\Phi(h_2(t))}{\Phi(-h_2(t))} \\
&= \frac{\Phi(h_1(t))}{d(t)\Phi(-h_2(t))} \qquad (21)\\
&= \frac{V_te^{r(T-t)}\Phi(-x_1(t))}{B\Phi(-x_2(t))}. \qquad (22)
\end{aligned}
$$

The relationship between the mean recovery rate and the European put price p_t at time t is given by

$$
\begin{aligned}
\mathrm{E}_t^Q[\delta(T)] &= \frac{\mathrm{E}_t^Q\left[\frac{B-(B-V_T)_+}{B}\right]-1\,\Pr(V_T\geq B)}{\Pr(V_T<B)} \\
&= \frac{\frac{B-p_te^{r(T-t)}}{B}-\Phi(h_2(t))}{\Phi(-h_2(t))} \\
&= \frac{1-p_t\frac{e^{r(T-t)}}{B}-\Phi(h_2(t))}{\Phi(-h_2(t))} \\
&= \frac{\Phi(-h_2(t))-p_t\frac{e^{r(T-t)}}{B}}{\Phi(-h_2(t))} \\
&= 1-\frac{e^{r(T-t)}}{B\Phi(-h_2(t))}p_t. \qquad (23)
\end{aligned}
$$

The distribution of $\ln(\frac{V_T}{B})$ at time t follows

$$\ln \tfrac{V_T}{B} \sim N\left(\ln \tfrac{V_t}{B} + \left(r - \frac{\sigma^2}{2}\right)(T-t), \sigma\sqrt{T-t}\right). \tag{24}$$

At time t, the probability density function for the recovery rate $w = \delta(T) = \frac{V_T}{B} \mid_{V_T<B}$ is

$$f_{\delta(T)}(w) = \frac{1}{\sqrt{2\pi(T-t)}\sigma w \Phi(-x_2(t))} e^{-\frac{\left(\ln\left(\frac{wB}{V_t}\right) - \left(r - \frac{\sigma^2}{2}\right)(T-t)\right)^2}{2\sigma^2(T-t)}} \tag{25}$$

where $0 < w < 1$.

The mean recovery rate is usually a function of the seniority of the bond. Assume that a firm issues bonds of five seniority classes: senior secured, senior unsecured, senior subordinated, subordinated, and junior subordinated (Carty and Lieberman, 1996). The five seniority class each represents proportions $p_1, p_2, ...p_5$ of the debt B where $\sum_i p_i = 1$. Similar to Madan and Unal (1995) and Lando (1997), we assume that the strick priority rule is applied: senior claimants are fully paid before any money distributed to junior claimants. The recovery rate for these five seniority classes are denoted by $\delta_1(T), \delta_2(T), ..., \delta_5(T)$. Thus recovery rates for each seniority classes are

$$\begin{aligned}
\delta_1(T) &= \min\left(\frac{\delta(T)}{p_1}, 1\right), \\
\delta_2(T) &= \min\left(\frac{(\delta(T) - p_1)_+}{p_2}, 1\right), \\
\delta_3(T) &= \min\left(\frac{(\delta(T) - p_1 - p_2)_+}{p_3}, 1\right), \\
\delta_4(T) &= \min\left(\frac{(\delta(T) - p_1 - p_2 - p_3)_+}{p_4}, 1\right), \\
\delta_5(T) &= \frac{(\delta(T) - p_1 - p_2 - p_3 - p_4)_+}{p_5}.
\end{aligned} \tag{26}$$

The probability density functions of the recovery rates are:

$$f_{\delta_1(T)}(w) = \begin{cases} \frac{1}{\sqrt{2\pi(T-t)}\sigma w \Phi(-x_2(t))} e^{-\frac{\left(\ln\left(\frac{wp_1B}{V_t}\right) - (r-\sigma^2/2)(T-t)\right)^2}{2\sigma^2(T-t)}}, & 0 < w < 1 \\ \int_{p_1}^{1} \frac{1}{\sqrt{2\pi(T-t)}\sigma u \Phi(-x_2(t))} e^{-\frac{\left(\ln\left(\frac{uB}{V_t}\right) - (r-\sigma^2/2)(T-t)\right)^2}{2\sigma^2(T-t)}} du, & w = 1 \\ 0, & w = 0, \end{cases} \tag{27}$$

$$f_{\delta_5(T)}(w) = \begin{cases} \frac{1}{\sqrt{2\pi(T-t)}\sigma(w + p_5^{-1} - 1)\Phi(-x_2(t))} e^{-\frac{\left(\ln\left(\frac{(p_1+p_2+p_3+p_4+p_5w)B}{V_t}\right) - (r-\sigma^2/2)(T-t)\right)^2}{2\sigma^2(T-t)}}, & 0 < w < 1 \\ \int_0^{p_4} \frac{1}{\sqrt{2\pi(T-t)}\sigma u \Phi(-x_2(t))} e^{-\frac{\left(\ln\left(\frac{uB}{V_t}\right) - (r-\sigma^2/2)(T-t)\right)^2}{2\sigma^2(T-t)}} du, & w = 0 \\ 0, & w = 1, \end{cases}$$

and

$$f_{\delta_k(T)}(w) = \begin{cases} \frac{p_k}{\sqrt{2\pi(T-t)}\sigma(p_1+\ldots+p_{k-1}+p_k w)\Phi(-x_2(t))} e^{-\frac{\left(\ln\left(\frac{(p_1+\ldots+p_{k-1}+p_k w)B}{V_0}\right)-(r-\sigma^2/2)(T-t)\right)^2}{2\sigma^2(T-t)}}, & 0 < w < 1 \\ \int_{p_k}^{1} \frac{1}{\sqrt{2\pi(T-t)}\sigma u \Phi(-x_2(t))} e^{-\frac{\left(\ln\left(\frac{uB}{V_t}\right)-(r-\sigma^2/2)(T-t)\right)^2}{2\sigma^2(T-t)}} du, & w = 1 \\ \int_{0}^{p_{k-1}} \frac{1}{\sqrt{2\pi(T-t)}\sigma u \Phi(-x_2(t))} e^{-\frac{\left(\ln\left(\frac{uB}{V_t}\right)-(r-\sigma^2/2)(T-t)\right)^2}{2\sigma^2(T-t)}} du, & w = 0 \end{cases}$$

for $k = 2, 3, 4$.

1.3 Unifying the Merton's structural model with a reduced-form model

In this section, we show that a Merton (1974) structural model is equivalent to a discrete reduced-form model. If the parameters of the value of a firm process are given, the probability of default can be derived from equation (9) and recovery rate from equation (21) or equation (22). By pointing out the default probability and mean recovery rate in the Merton's (1974) credit spread formula, the credit spread formula can be rewritten as the price of a contingent claim under risk-neutral valuation paying one if there is no default and paying the recovery rate if default happens at maturity.

In the other direction, we can start with a reduced-form model under risk-neutral valuation with the assumption of independence between default probability and its recovery. The corresponding value of the firm process defines the default probability and mean recovery rate. From the yield spread formula equation (61) or equation (29) for a reduced-form model, we can derive the Merton's (1974) credit spread formula. First we start with Merton's credit spread formula equation (4):

$$\begin{aligned} R(t,T) - r &= -\frac{1}{T-t} \ln\left\{\Phi(h_2(t)) + \frac{1}{d}\Phi(h_1(t))\right\} && (28) \\ &= -\frac{1}{T-t} \ln\left\{1 - \Phi(-h_2(t)) + \frac{1}{d}\Phi(h_1(t))\right\} \\ &= -\frac{1}{T-t} \ln\left\{1 - \Phi(-h_2(t))\left[1 - \frac{\Phi(h_1(t))}{d\Phi(-h_2(t))}\right]\right\} \\ &= -\frac{1}{T-t} \ln\left\{1 - \Pr\{V_T < B\}\left[1 - \mathrm{E}_t^Q[\delta(T)]\right]\right\} && (29) \\ &= -\frac{1}{T-t} \ln\left\{1 - \mathrm{E}_t^Q[I_{(V_T<B)}]\left[\mathrm{E}_t^Q[1-\delta(T)]\right]\right\} && (30) \\ &= -\frac{1}{T-t} \ln \mathrm{E}_t^Q\left[1 - I_{(\tau=T)}\left[1-\delta(T)\right]\right] && (31) \\ &= -\frac{1}{T-t} \ln \mathrm{E}_t^Q\left[1 - I_{(\tau=T)} + I_{(\tau=T)}\delta(T)\right] \\ &= -\frac{1}{T-t} \ln \mathrm{E}_t^Q\left[1 I_{(\tau>T)} + \delta(T) I_{(\tau=T)}\right]. && (32) \end{aligned}$$

After rearrangement, we have

$$R(t,T)(T-t) \quad = \quad r(T-t) - \ln \mathrm{E}_t^Q \left[I_{(\tau>T)} + \delta(T) I_{(\tau=T)} \right].$$

After taking logarithms, we have

$$\begin{aligned} e^{-R(T-t)} &= e^{-r(T-t)} \mathrm{E}_t^Q \left[I_{(\tau>T)} + \delta(T) I_{(\tau=T)} \right] \\ &= \mathrm{E}_t^Q \left[e^{-r(T-t)} [I_{(\tau>T)} + \delta(T) I_{(\tau=T)}] \right]. \end{aligned} \tag{33}$$

Finally, we have

$$V(t,T) = \mathrm{E}_t^Q \left[\frac{B(t)}{B(T)} \left[I_{(\tau>T)} + \delta(T) I_{(\tau=T)} \right] \right]. \tag{34}$$

This shows that the Merton's model equivalently defines a reduced form model. From a reduced-form yield spread formula equation (29), we can derive the Merton's (1974) risk premium.

Note: For the structural model, we do not assume the independence between the default probability and the recovery rate to derive the reduced-form models since we do not need any independence assumption in equation (30) to derive equation (31). However, if we go in the opposite direction we need an independence assumption to go from equation (31) to equation (30).

2 Reduced-form credit risk models

2.1 The first passage time and intensity process

We assume the existence of a Q-measure probability regarding the uncertainty under risk-neutral measure. As a result, there is no arbitrage opportunity. All random variables are defined on a filtered probability space on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), Q)$ for uncertainty under Q-measure probability. Let the time of default (a stopping time) be denoted as τ. Specifically in a structural model, if the time of default can be any time before the underlying bond's maturity, then the time of default is defined as $\tau \equiv \inf\{t : V_t < B\}$. By the Girsanov's theorem and the strong Markov property of the Brownian, we have

$$\tau \equiv \inf\{t : V_t < B\} = \inf\{t : V_t \leq B\}. \tag{35}$$

A right-continuous counting process is defined as $N(t) = 1_{\{\tau \leq t\}}$. Let $F_\tau(t)$, $S_\tau(t)$, and $f_\tau(t)$ denote the distribution function, survival function and the probability density function of τ respectively,

$$S_\tau(t) = 1 - F_\tau(t) = 1 - \Pr\{\tau \leq t\} \tag{36}$$

and

$$f_\tau(t) = \frac{d}{dt} F_\tau(t). \tag{37}$$

An intensity function is defined as

$$\mu_\tau(t) = \lim_{h\to 0}\frac{1}{h}\Pr\{N(t+h)-N(t)=1 \mid N(t)=0\} \tag{38}$$
$$= \frac{f_\tau(t)}{1-F_\tau(t)} \tag{39}$$
$$= \frac{-S'_\tau(t)}{S_\tau(t)}. \tag{40}$$

An adapted process $\{[1-N(t)]\mu_\tau(t) : t > 0\}$ is called an intensity process which is assumed to be non-negative and predictable with $\int_0^t \mu_\tau(s)ds < \infty$ for all $t > 0$ almost surely. Artzner and Delbaen (1995) showed that the existence of an intensity process under P-measure guarantees the existence of an intensity process under any equivalent measure such as a Q-measure probability. The distribution function and survival function of time of default can be expressed as a function of the intensity function as

$$F_\tau(t) = 1 - S_\tau(t) = 1 - e^{-\int_0^t \mu_\tau(s)ds}. \tag{41}$$

The intensity is also called the hazard rate, failure rate function, or force of default with a continuous compensator $A(t) = \int_0^t \mu_s ds$ which is increasing and predictable with the property that $N(t) - A(t)$ is a martingale.

Two actuarial symbols are imported here. Assume a company has survived x years and its future life time is $\tau(x)$, two conditional probabilities are defined as:

$$_tp_x = \Pr\{\tau(x) > t \mid \tau(x) > 0\}, \tag{42}$$

and

$$_tq_x = \Pr\{\tau(x) \le t \mid \tau(x) > 0\}. \tag{43}$$

We also have:

$$_tp_x = 1 - {}_tq_x = e^{-\int_0^t \mu_\tau(x+s)ds} \tag{44}$$

The frequency of a default event is usually assumed to be Poisson distributed. We begin with considering a discrete default process for a single entity. Assume the default probability of a company in a short time period is q. Default events do not occur very often so that we tend to consider q as a small number. The probability generating function of counting process N is

$$\begin{aligned} P(z) &= (1-q) + qz \\ &= \exp\{\ln[1+q(z-1)]\} \\ &= \exp[q(z-1) + O(q^2)] \\ &\cong \exp[q(z-1)] \end{aligned}$$

which is a Poisson probability generating function. For a short time period when $O(q^2)$ is negligible, the default frequency has approximately a Poisson distribution.

Let $N(t)$ represent the number of total default events up to time t, so that $N(t) = \sum_{i=0}^{n} N_i(t)$ where $N_i(t) = 1_{\{\tau_i \le t\}}$ represents for the one-jump process for entity i at time t. Assume the counting process $N(t)$ is nondecreasing in t with independent increments and the probability of default in a short time period is proportional to the length of time or mathematically:

(1)$N(0) = 0, \qquad N(t) \in \{0, 1, 2, \ldots\}$,
(2) If $s < t$ then $N(s) \le N(t)$,

(3) $$\Pr\{N(t+h) = n+m \mid N(t) = n\} = \begin{cases} 1 - \mu(t)h + o(h) & m = 0 \\ \mu(t)h + o(h) & m = 1 \\ o(h) & m > 1, \end{cases}$$

(4)$\{N(t), t \ge 0\}$ has independent increments.

We can conclude that the number of default events $N(t)$ follow a time-dependent nonhomogeneous Poisson process with intensity function $\mu(t), t \ge 0$. If we take $n = 0$ and $m = 1$, we have:

$$\lim_{h \to 0} \frac{1}{h} \Pr\{N(t+h) - N(t) = 1 \mid N(t) = 0\} = \lim_{h \to 0} \frac{\mu(t)h + o(h)}{h} = \mu(t). \tag{45}$$

2.2 Recovery scheme

The recovery rate is defined as the ratio of debt recovered once a default event happens. It is denote by $\delta(\tau)$ if a default event can happen any time before the maturity and is denoted by $\delta(T)$ if a default event can happen only at maturity time T. There are three major recovery schemes modelling credit risk: recovery of par value (RPV), recovery of treasury value (RTV), and recovery of market value (RMV).

To illustrate the three recovery schemes, we start with giving some notations. Let $P(t,T)$ represent the time-t price of a default-free zero-coupon bond paying one dollar at maturity time T where $0 \le t \le T < \infty$ and $V(t,T)$ represent the time-t price of a defaultable corporate zero-coupon bond paying one dollar at maturity time T. The yields to maturity for the default-free bond and defaultable bond are denoted as $r(t,T)$ and $R(t,T)$ respectively. The bond prices can be written as

$$P(t,T) = \exp\left[-r(t,T)(T-t)\right] \tag{46}$$

and

$$V(t,T) = \exp\left[-R(t,T)(T-t)\right]. \tag{47}$$

A time-t value of a bank account process or the value of one dollar accumulated from time 0 to time t is defined as

$$B(t) = \exp\left[\int_0^t r(s)ds\right] \tag{48}$$

where $r(s)$ is the spot rate at time-s.

When a default event happens and the firm is immediately liquidated at the time of default to pay the bond holders a fraction payment of face amount, the scheme is called recovery of par value (RPV). By assuming the existence of an Q-measure probability, a risky bond price under a RPV scheme can be written as

$$V(t,T) \quad = \quad \mathrm{E}_t^Q \left[\frac{B(t)}{B(\tau)} \delta(\tau) 1_{(\tau \leq T)} + \frac{B(t)}{B(T)} 1_{(\tau > T)} \right]. \tag{49}$$

When a default event happens and the firm is liquidated later at the time of the maturity of the bond to pay the bond holders a fraction payment of face amount, the scheme is called recovery of treasury value (RTV). The remaining value of the defaulted firm does not earn any interest during the period from default time to maturity. If it does earn interest, the scheme is equivalent to a recovery of par value scheme. A risky bond price for a RTV scheme under risk-neutral valuation can be written as

$$V(t,T) \quad = \quad \mathrm{E}_t^Q \left[\frac{B(t)}{B(T)} \left[\delta(\tau) 1_{(\tau \leq T)} + 1_{(\tau > T)} \right] \right]. \tag{50}$$

It is almost impossible that a firm can be liquidated immediately at the default time. The time of liquidation is usually at a later time and can be a random variable. In this paper, we adopt RTV scheme where the liquidation time is fixed as the time of maturity of the bond.

If a default event happens and the bond holders receive a fraction of the pre-default market value of bond from the liquidation of the firm at time of default, we call this recovery scheme the recovery of market value (RMV). The price of a risky bond can be written as

$$V(t,T) \quad = \quad \mathrm{E}_t^Q \left[\frac{B(t)}{B(\tau)} (1 - L_\tau) V(\tau-, T) 1_{(\tau \leq T)} + \frac{B(t)}{B(T)} 1_{(\tau > T)} \right] \tag{51}$$

where L_τ represents the expected fractional loss in market value at time of default. Duffie and Singleton (1995) showed that the price of a risky bond can be generally written as

$$V(t,T) = \mathrm{E}_t^Q \left[\exp \left(- \int_t^T R(s) ds \right) X \right] \tag{52}$$

where X is the promised payment among maturity in the event of no default and is 1 for a zero-coupon bond. An instantaneous credit spread or an instantaneous short-rate spread can be expressed as the product of the mean loss ratio and the intensity of default as

$$R(s) - r(s) = (1 - L_s)\mu_s \tag{53}$$

where $R(s)$ represents the default-adjusted short rate at time s.

When a firm issues bonds of several seniorities, we say that the strict priority rule is applied if the senior claimants are fully paid before any money distributed to the junior claimants. We will show that the distribution of jump can be derived through comparison of the bonds prices of different seniorities in a jump-diffusion process.

2.3 Risk neutral valuation and the yield spread for a reduced-form model

Based on the model of Jarrow and Turnbull (1995), we assume the financial market is frictionless with a finite time horizon. By assuming the existence of an Q-measure probability, the arbitrage opportunity is excluded. The price of a riskless bond can be written as

$$P(t,T) = \mathrm{E}_t^Q\left[\frac{B(t)}{B(T)}\right] = \mathrm{E}_t^Q\left[\exp\left(-\int_t^T r(s)ds\right)\right]. \tag{54}$$

Based on a RTV scheme and the assumption of the existence of a Q-measure probability, the price of a risky bond can be written as

$$\begin{aligned} V(t,T) &= \mathrm{E}_t^Q\left[\frac{B(t)}{B(T)}\left[\delta(\tau)1_{(\tau\le T)} + 1_{(\tau>T)}\right]\right] & (55)\\ &= \mathrm{E}_t^Q\left[\frac{B(t)}{B(T)}\left\{1-[1-\delta(\tau)]1_{(\tau\le T)}\right\}\right]. & (56)\end{aligned}$$

If the default-free spot rates and the default process are independent, the price of a risky bond is

$$\begin{aligned} V(t,T) &= \mathrm{E}_t^Q\left[\frac{B(t)}{B(T)}\right]\mathrm{E}_t^Q\left[1-\{1-\delta(\tau)\}1_{(\tau\le T)}\right] & (57)\\ &= P(t,T)\left\{1-\mathrm{E}_t^Q\left[\{1-\delta(\tau)\}1_{(\tau\le T)}\right]\right\}, & (58)\end{aligned}$$

and it can be written as a function of the yields to maturity as

$$e^{-R(t,T)(T-t)} = e^{-r(t,T)(T-t)}\left\{1-\mathrm{E}_t^Q\left[\{1-\delta(\tau)\}1_{(\tau\le T)}\right]\right\}. \tag{59}$$

Then, the credit spread, risk premium, or the yield spread is

$$R(t,T)-r(t,T) = -\frac{1}{T-t}\ln\left\{1-\mathrm{E}_t^Q\left[\{1-\delta(\tau)\}1_{(\tau\le T)}\right]\right\}. \tag{60}$$

In the case that the recovery rate is independent of the default probability, the yield spread is

$$\begin{aligned} R(t,T)-r(t,T) &= -\frac{1}{T-t}\ln\left\{1-\{1-\mathrm{E}_t^Q[\delta(\tau)]\}\Pr\{\tau\le T\}\right\} & (61)\\ &= -\frac{1}{T-t}\ln\left\{1-\{1-\mathrm{E}_t^Q[\delta(\tau)]\}_{T-t}\tilde{q}_t\right\} & (62)\end{aligned}$$

where ${}_{T-t}\tilde{q}_t$ represents the Q-measure probability that a firm will default before time T given it has not defaulted at time t.

The product of the mean loss rate and the default probability can be written as a function of bond prices. If the default-free spot rates independent of default process, the equation (56) can be written as

$$
\begin{aligned}
V(t,T) &= \mathrm{E}_t^Q\left[\frac{B(t)}{B(T)}\left\{1-[1-\delta(\tau)]1_{(\tau\le T)}\right\}\right] \\
&= \mathrm{E}_t^Q\left[\frac{B(t)}{B(T)}\right]-\mathrm{E}_t^Q\left[\frac{B(t)}{B(T)}\{1-\delta(\tau)\}1_{(\tau\le T)}\right] \\
&= P(t,T)-P(t,T)\mathrm{E}_t^Q\left[\{1-\delta(\tau)\}1_{(\tau\le T)}\right].
\end{aligned}
$$

After rearrangement, it becomes

$$
\mathrm{E}_t^Q\left[\{1-\delta(\tau)\}1_{(\tau\le T)}\right] = \frac{P(t,T)-V(t,T)}{P(t,T)}. \tag{63}
$$

If the recovery rate is independent of the default probability, the product of the mean loss rate and the default probability is

$$
\left\{1-\mathrm{E}_t^Q[\delta(\tau)]\right\}\Pr\{\tau\le T\} = \frac{P(t,T)-V(t,T)}{P(t,T)}, \tag{64}
$$

or

$$
1-\left\{1-\mathrm{E}_t^Q[\delta(\tau)]\right\}\Pr\{\tau\le T\} = \frac{V(t,T)}{P(t,T)}. \tag{65}
$$

There are two unknown; $\Pr\{\tau\le T\}$ and $\mathrm{E}_t^Q[\delta(\tau)]$, in one equation. As long as one of the them is known, the term structure of default for the future can be derived. At least one more assumption is necessary to separate the mean recovery rate and default probability. Different choices of this assumption lead to different models.

2.4 Discrete time term structure and the forward spread

In a discrete time model, the forward rate for a riskless bond is defined as

$$
f(t,T)\equiv-\ln\left\{\frac{P(t,T+1)}{P(t,T)}\right\}, \tag{66}
$$

and the forward rate for a risky zero-coupon bond is defined as

$$
f^i(t,T)\equiv-\ln\left\{\frac{V(t,T+1)}{V(t,T)}\right\}. \tag{67}
$$

If the default-free spot rates and the default process are independent in a RTV scheme, from equation (58), the forward rate for a risky zero-coupon bond can be written as

$$
\begin{aligned}
f^i(t,T) &= -\ln\left\{\frac{V(t,T+1)}{V(t,T)}\right\} \\
&= -\ln\left\{\frac{P(t,T+1)\left\{1-\mathrm{E}_t^Q\left[\{1-\delta(\tau)\}1_{(\tau\le T+1)}\right]\right\}}{P(t,T)\left\{1-\mathrm{E}_t^Q\left[\{1-\delta(\tau)\}1_{(\tau\le T)}\right]\right\}}\right\} \\
&= f(t,T)-\ln\left\{\frac{1-\mathrm{E}_t^Q\left[\{1-\delta(\tau)\}1_{(\tau\le T+1)}\right]}{1-\mathrm{E}_t^Q\left[\{1-\delta(\tau)\}1_{(\tau\le T)}\right]}\right\}.
\end{aligned} \tag{68}
$$

If the recovery rate is independent of the default probability, the credit spread or the forward spread is given by

$$
\begin{aligned}
f^i(t,T) - f(t,T) &= -\ln\left\{\frac{1-\left(1-\mathrm{E}_t^Q[\delta(\tau;T+1)]\right)\ {}_{T+1-t}\tilde{q}_t}{1-\left(1-\mathrm{E}_t^Q[\delta(\tau;T)]\right)\ {}_{T-t}\tilde{q}_t}\right\} \\
&= \ln\left\{\frac{1-\left(1-\mathrm{E}_t^Q[\delta(\tau;T)]\right)\ {}_{T-t}\tilde{q}_t}{1-\left(1-\mathrm{E}_t^Q[\delta(\tau;T+1)]\right)\ {}_{T+1-t}\tilde{q}_t}\right\} \qquad (69)
\end{aligned}
$$

where $\mathrm{E}_t^Q[\delta(\tau;T)]$ represents the Q-measure mean recovery rate if the maturity is time T.

If the values of $f^i(t,T)$ and $f(t,T)$ on all Ts are available from the market, then the credit spreads can be derived. By letting $T=t$, we have the spread for spot rate

$$R(t) - r(t) = \ln\left\{\frac{1}{1-\left(1-\mathrm{E}_t^Q[\delta(\tau;1)]\right)\ {}_1\tilde{q}_t}\right\}. \qquad (70)$$

If the Q-measure mean recovery rate $\mathrm{E}_t^Q\{\delta(T)\}$ is known, the conditional Q-measure default probability for the first period is:

$${}_1\tilde{q}_t = \frac{1-e^{-(R(t)-r(t))}}{1-\mathrm{E}_t^Q[\delta(\tau;1)]}. \qquad (71)$$

The rest of the default probabilities for the future can be derived directly from

$${}_{T-t}\tilde{q}_t = \frac{1-\frac{V(t,T)}{P(t,T)}}{1-\mathrm{E}_t^Q[\delta(\tau;T)]}. \qquad (72)$$

Expressed in terms of the intensity, the default probability can be written as

$$1-e^{-\int_t^T \tilde{\mu}(s)ds} = \frac{1-\frac{V(t,T)}{P(t,T)}}{1-\mathrm{E}_t^Q[\delta(T)]}. \qquad (73)$$

After rearrangement, we have

$$-\int_t^T \tilde{\mu}(s)ds = \ln\left\{\frac{V(t,T)}{P(t,T)} - \mathrm{E}_t^Q(\delta(T))\right\} - \ln\{1-\mathrm{E}_t^Q(\delta(T))\}. \qquad (74)$$

Since this is in a discrete time environment, t takes integer values. If the intensity of default is constant within each period, then the intensity at time t is

$$\tilde{\mu}(t) = \ln\{1-\mathrm{E}_t^Q[\delta(n+1)]\} - \ln\left\{\frac{V(n,n+1)}{P(n,n+1)} - \mathrm{E}_t^Q[\delta(n+1)]\right\},\ n \le t < n+1. \qquad (75)$$

2.5 Continuous time term structure and the instantaneous forward spread

In a continuous time model, the forward rate for a riskless zero-coupon bond is defined as

$$f(t,T) \equiv \frac{-\partial}{\partial T} \ln P(t,T), \tag{76}$$

and the instantaneous forward rate for a risky zero-coupon bond is defined as

$$f^i(t,T) \equiv \frac{-\partial}{\partial T} \ln V(t,T). \tag{77}$$

If the default-free spot rates and the default process are independent in a RTV scheme, from equation (58), the instantaneous forward rate for a risky zero-coupon bond can be written as

$$\begin{aligned}
f^i(t,T) &= \frac{-\partial}{\partial T} \ln V(t,T) \\
&= \frac{-\partial}{\partial T} \ln \left\{ P(t,T) \left\{ 1 - \mathrm{E}_t^Q \left[\{1-\delta(\tau)\} 1_{(\tau \le T)} \right] \right\} \right\} \\
&= \frac{-\partial}{\partial T} \left\{ \ln P(t,T) + \ln \left\{ 1 - \mathrm{E}_t^Q \left[\{1-\delta(\tau)\} 1_{(\tau \le T)} \right] \right\} \right\} \\
&= f(t,T) + \frac{-\partial}{\partial T} \ln \left\{ 1 - \mathrm{E}_t^Q \left[\{1-\delta(\tau)\} 1_{(\tau \le T)} \right] \right\} \\
&= f(t,T) + \frac{\frac{\partial}{\partial T} \mathrm{E}_t^Q \left[\{1-\delta(\tau)\} 1_{(\tau \le T)} \right]}{1 - \mathrm{E}_t^Q \left[\{1-\delta(\tau)\} 1_{(\tau \le T)} \right]}.
\end{aligned} \tag{78}$$

If the recovery rate is independent of the default probability, the credit spread or the instantaneous forward spread is given by

$$f^i(t,T) - f(t,T) = \frac{\frac{\partial}{\partial T} \left[\{1 - \mathrm{E}_t^Q[\delta(\tau)]\} \Pr\{\tau \le T\} \right]}{1 - \{1 - \mathrm{E}_t^Q[\delta(\tau)]\} \Pr\{\tau \le T\}}. \tag{79}$$

In the case where $\mathrm{E}_t^Q[\delta(\tau)]$ is not a function of T, the instantaneous forward spread is

$$f^i(t,T) - f(t,T) = \frac{\{1 - \mathrm{E}_t^Q[\delta(\tau)]\} \frac{\partial}{\partial T} \Pr\{\tau \le T\}}{1 - \{1 - \mathrm{E}_t^Q[\delta(\tau)]\} \Pr\{\tau \le T\}} \tag{80}$$

$$\begin{aligned}
&= \frac{\{1 - \mathrm{E}_t^Q[\delta(\tau)]\} \frac{\partial}{\partial T} \{1 - e^{-\int_t^T \tilde{\mu}(s) ds}\}}{1 - \{1 - \mathrm{E}_t^Q[\delta(\tau)]\}\ {}_{T-t}\tilde{q}_t} \\
&= \frac{\{1 - \mathrm{E}_t^Q[\delta(\tau)]\} e^{-\int_t^T \tilde{\mu}(s) ds} \tilde{\mu}(T)}{1 - \{1 - \mathrm{E}_t^Q[\delta(\tau)]\}\ {}_{T-t}\tilde{q}_t} \\
&= \frac{\{1 - \mathrm{E}_t^Q[\delta(\tau)]\} (1 - {}_{T-t}\tilde{q}_t) \tilde{\mu}(T)}{1 - \{1 - \mathrm{E}_t^Q[\delta(\tau)]\}\ {}_{T-t}\tilde{q}_t} \\
&= \tilde{\mu}(T) \frac{1 - \mathrm{E}_t^Q[\delta(\tau)] - \{1 - \mathrm{E}_t^Q[\delta(\tau)]\}\ {}_{T-t}\tilde{q}_t}{1 - \{1 - \mathrm{E}_t^Q[\delta(\tau)]\}\ {}_{T-t}\tilde{q}_t}
\end{aligned}$$

$$= \tilde{\mu}(T) \left\{ 1 - \frac{\mathrm{E}_t^Q[\delta(\tau)]}{1 - \{1 - \mathrm{E}_t^Q[\delta(\tau)]\}\ {}_{T-t}\tilde{q}_t} \right\} \tag{81}$$

$$= \tilde{\mu}(T) \left\{ 1 - \frac{\mathrm{E}_t^Q[\delta(\tau)]}{1 - \{1 - \mathrm{E}_t^Q[\delta(\tau)]\} \Pr\{\tau \le T\}} \right\} \tag{82}$$

where equation (80) is the same result as Jarrow and Turnbull (1997), equation (23). If we substitute equation (65) into equation (82), we find that the instantaneous forward spread is simply

$$\begin{aligned} f^i(t,T) - f(t,T) &= \tilde{\mu}(T)\left\{1 - \frac{\mathrm{E}_t^Q[\delta(\tau)]}{\frac{V(t,T)}{P(t,T)}}\right\} \\ &= \tilde{\mu}(T)\left\{1 - \mathrm{E}_t^Q[\delta(\tau)]\frac{P(t,T)}{V(t,T)}\right\}. \end{aligned} \tag{83}$$

If the mean recovery rate is known, the intensity process is given by

$$\tilde{\mu}(T) = \frac{f^i(t,T) - f(t,T)}{1 - \mathrm{E}_t^Q[\delta(\tau)]\frac{P(t,T)}{V(t,T)}}. \tag{84}$$

Jarrow and Turnbull (1997) shows that , the instantaneous forward spread degenerates into a spot rate spread as $T \to t$. In a Duffie and Singleton (1995) model under a RMV scheme, the amount recovered at default is $(1 - L_\tau)V(\tau_-, T)$. The relation between the recovery rate in a RTV scheme and the recovery rate in a RMV scheme is

$$\mathrm{E}_t^Q[\delta(\tau)] = (1 - L_\tau)V(\tau_-, T). \tag{85}$$

In our formula,

$$\begin{aligned} f^i(t,t_+) - f(t,t_+) &= R(t_+) - r(t_+) \\ &= \tilde{\mu}(t_+)\left\{1 - \mathrm{E}_t^Q[\delta(t_+)]\frac{P(t,t_+)}{V(t,t_+)}\right\} \\ &= \tilde{\mu}(t_+)\left\{1 - \frac{\mathrm{E}_t^Q[\delta(t_+)]}{V(t,t_+)}\right\} \\ &= \tilde{\mu}(t_+)\left\{1 - \frac{(1 - L_{t_+})V(t,t_+)}{V(t,t_+)}\right\} \\ &= \tilde{\mu}(t_+)L_{t_+} \end{aligned} \tag{86}$$

which is spot-rate spread formula derived by Duffie and Singleton (1995). However, the instantaneous forward spread generally does not equal the spot-rate spread except at the time right before the time of maturity of the bond.

3 A jump-diffusion structural model

4 Introduction

Merton (1974) assumes that a default event can happen only through a diffusion process when the value of the firm falls below its underlying liability. A sudden jump of the value of the company can not be described through this model. In order to capture these two kinds of default events which are caused either by a jump or a diffusion process, Zhou (1997) adopts the Merton's (1976) jump-diffusion process to model credit risk.

To build a jump-diffusion model, we need the following assumption:

1. The market is perfect and frictionless. There are many investors who can sell, short and buy as much as they want in continuous time.
2. The term structure for interest rate is flat. The interest is fixed all time no matter borrowing or lending
3. The Modigliani-Miller theorem that the value of the firm is invariant to its capital structure.
4. In a jump-diffusion process proposed by Merton (1976), the capital asset pricing model(CAPM) holds for equilibrium returns and the jump components represent nonsystematic risk which has zero beta.
5. There is a constant threshold B that default event happens whenever the value of the firm falls below this threshold. The debt holders received cash in according its bond's seniority or we say the strick priority rule is applied.

The value of a company consists of a diffusion process which represent the systematic risk and is correlated with the market and a jump process with its compensator which represents nonsystematic risk and is uncorrelated with the market:

$$dV_t = \alpha V_t dt + \sigma V_t dZ_t + (J_{N_t} dN_t - \lambda \mu_J dt) V_t \tag{87}$$

$$= (\alpha - \lambda \mu_J) V_t dt + \sigma V_t dZ_t + V_t J_{N_t} dN_t \tag{88}$$

where

V_t: value of the firm at time t,
α: instantaneous expected rate of return,
σ^2: instantaneous variance of return,
Z_t: a standard Brownian motion,
N_t: total number of jumps up to time t,
J_{N_t}: jump size as a proportion of V_t,

and $\{N_t; t \geq 0\}$ is a counting process which follows a Poisson process with parameter λ. The jump amplitudes $J_1, J_2, \cdots$ are assumed to be independently and identically distributed positive random variables whose moment generating function exists with mean μ_J and variance σ_J^2. The mean value of the instantaneous change of the value of a firm is

$$\mathrm{E}[dV_t] = \alpha V_t dt + \sigma V_t \mathrm{E}[dZ_t] + (\mathrm{E}[J_{N_t} dN_t] - \lambda \mu_J dt) V_t$$

$$
\begin{aligned}
&= \alpha V_t dt + \sigma V_t 0 + (\lambda\mu_J dt - \lambda\mu_J dt)V_t \\
&= \alpha V_t dt. \qquad (89)
\end{aligned}
$$

The Poisson process does not play any role in deciding the expected return on the value of the firm but it allows the possibility of instantaneous jump of the value of the firm. The path of the value of the firm is different from pure diffusion process and the default probability and recovery rate are also different. If we divide both sides by V_t, we have,

$$
dV_t/V_t = (\alpha - \lambda\mu_J)dt + \sigma dZ_t + J_{N_t}dN_t. \qquad (90)
$$

V_t is the value of a company and is assumed to be a nonnegative number. The percentage of change $\frac{dV_t}{V_t}$ at most can go down to -1 and can go up to infinity. The range of the jump amplitude J can only take values in $[-1, \infty]$. J equals a nonnegative random variable minus 1. Assume that Y is a nonnegative random variable and $J_i = Y_i - 1$ for $i = 1$ to $N(t)$, then J can fall in $[-1, \infty]$. Y is an impulse function takes value 1 when there is no jump and takes values from 0 to ∞ other than 1 if there is a jump. Let

$$
Y(t) = \prod_{i=1}^{N(t)} Y_i = \prod_{i=1}^{N(t)} (J_i + 1).
$$

From Itô's lemma, we have

$$
\begin{aligned}
d\ln V_t &= \left[\frac{1}{V_t}(\alpha V_t - \lambda\mu_J V_t) + \frac{1}{2} \times (-\frac{1}{V_t^2})(\sigma V_t)^2\right] dt + \sigma dZ_t \\
&\quad +[\ln(V_t + V_t J_t) - \ln(V_t)]dN(t) \\
&= \left(\alpha - \frac{\sigma^2}{2} - \lambda\mu_J\right) dt + \sigma dZ_t + \ln(J_{N_t} + 1)dN(t) \qquad (91) \\
&= \left(\alpha - \frac{\sigma^2}{2} - \lambda\mu_J\right) dt + \sigma dZ_t + \ln Y_{N_t} dN(t). \qquad (92) \\
& \qquad (93)
\end{aligned}
$$

Equivalently,

$$
\begin{aligned}
\ln V_t &= \ln V_0 + \left(\alpha - \frac{\sigma^2}{2} - \lambda\mu_J\right) t + \sigma Z_t + \sum_{i=0}^{N_t} \ln Y_i \qquad (94) \\
&= \ln V_0 + \left(\alpha - \frac{\sigma^2}{2} - \lambda\mu_J\right) t + \sigma Z_t + \ln Y(t). \qquad (95)
\end{aligned}
$$

A derivative of the value of the firm $C(V_t, t)$ must follow another diffusion process as

$$
dC(V_t, t) = [\alpha_c - \lambda\mu_c]C(V_t, t)dt + \sigma_c C(V_t, t)dZ_c + J_{N_t}^C dN_t. \qquad (96)
$$

From Itô's lemma, we have

$$
\begin{aligned}
dC(V_t, t) &= \left[(\alpha V_t - \lambda\mu_J)\frac{\partial C(V_t, t)}{\partial V_t} + \frac{\partial C(V_t, t)}{\partial t} + \frac{1}{2}\sigma^2 V_t^2 \frac{\partial^2 C(V_t, t)}{\partial V_T^2}\right] dt \\
&\quad +\sigma V_t \frac{\partial C(V_t, t)}{\partial V_t} dZ_c + [C(V_t Y_{N_t}, t) - C(V_t, t)]dN_t. \qquad (97)
\end{aligned}
$$

A representative agent can invest in the firm, the option and the riskless asset. The value of the portfolio P has instantaneous proportional return as

$$\frac{dP}{P} = (\alpha_p - \lambda\mu_p)dt + \sigma_p dZ_p + J_p dN_t. \tag{98}$$

Following Merton's (1976) argument, if the jump components represent nonsystematic risk, the portfolio has zero beta. If the CAPM holds, the expected return on all zero-beta securities must equal riskless rate. We have

$$\alpha_p = r$$

and a integral-differential equation

$$\frac{1}{2}\sigma^2 V_t^2 \frac{\partial^2 C(V_t,t)}{\partial V_t^2} + (r-\lambda\mu_J)V_t \frac{\partial C(V_t,t)}{\partial V_t} + \frac{\partial C(V_t,t)}{\partial t} - rC(V_t,t) + \lambda E[C(V_tY,t) - C(V_t,t)] = 0. \tag{99}$$

The value of the firm thus follows

$$dV_t/V_t = (r-\lambda\mu_J)dt + \sigma dZ_t + J_{N_t} dN_t \tag{100}$$

and

$$d\ln V_t = \left(r - \frac{\sigma^2}{2} - \lambda\mu_J\right)dt + \sigma dZ_t + \ln Y_{N_t} dN(t). \tag{101}$$

Equivalently, we have

$$\ln V_t = \ln V_0 + \left(r - \frac{\sigma^2}{2} - \lambda\mu_J\right)t + \sigma Z_t + \sum_{i=1}^{N_t} \ln Y_i, \tag{102}$$

$$\tag{103}$$

and

$$V_t = V_0 \exp\left[\left(r - \frac{\sigma^2}{2} - \lambda\mu_J\right)t + \sigma Z_t\right] Y(t). \tag{104}$$

The time-0 expected value of $e^{-rt}V_t$ is

$$\begin{aligned}
\mathrm{E}[e^{-rt}V_t] &= \mathrm{E}\left[e^{-rt}V_0 e^{\left(r-\frac{\sigma^2}{2}-\lambda(\mu_Y-1)\right)t+\sigma Z_t} Y(t)\right] \\
&= V_0 e^{\left[-\frac{\sigma^2}{2}-\lambda(\mu_Y-1)\right]t} E\left[e^{\sigma Z_t}\right] E\left[\Pi_{i=1}^{N_t} Y_i\right] \\
&= V_0 e^{\left[-\frac{\sigma^2}{2}-\lambda(\mu_Y-1)\right]t} e^{\frac{\sigma^2}{2}t} \sum_{n=0}^{\infty} \mathrm{E}\left[\Pi_{i=1}^{n} Y_i\right] \frac{(\lambda t)^n e^{-\lambda t}}{n!} \\
&= V_0 e^{-\lambda\mu_Y t + \lambda t} \sum_{n=0}^{\infty} \mu_Y^n \frac{(\lambda t)^n e^{-\lambda t}}{n!} \\
&= V_0 e^{-\lambda\mu_Y t} \sum_{n=0}^{\infty} \frac{[\lambda\mu_Y t]^n}{n!} \\
&= V_0
\end{aligned} \tag{105}$$

which is equivalently saying that no arbitrage is allowed and the existence of the Q-measure probability.

4.1 Unifying structural jump-diffusion model with reduced-form models-constant jumps

We start with a simple case where the jump amplitude is a constant. Let

$$Y_i = J_i + 1 = s \quad 0 < s < 1, \quad i \in \{0, 1, 2...\}. \tag{106}$$

The means satisfy $\mu_J = \mu_Y - 1 = s - 1$. From equation (102), the value of the firm follows

$$\ln V_t = \ln V_0 + \left[r - \frac{\sigma^2}{2} - \lambda(s-1)\right] t + \sigma Z_t + N_t \ln s.$$

Conditioned on the number of jump, the distribution of the value of the firm follows

$$\ln V_t \mid_{N_t=n} \sim N\left\{\ln V_0 + \left[r - \frac{\sigma^2}{2} - \lambda(s-1)\right] t + n \ln s, \sqrt{\sigma^2 t}\right\}. \tag{107}$$

As time shifts from 0 to the current time t, the probability of default at bond's maturity is given by

$$\begin{aligned} \Pr(\tau = T) &= \sum_{n=0}^{\infty} \frac{[\lambda(T-t)]^n e^{-\lambda(T-t)}}{n!} \Phi\left(-\frac{\ln \frac{V_t}{B} + [r - \frac{\sigma^2}{2} - \lambda(s-1)](T-t) + n \ln s}{\sqrt{\sigma^2(T-t)}}\right) \\ &= \sum_{n=0}^{\infty} \frac{[\lambda(T-t)]^n e^{-\lambda(T-t)}}{n!} \Phi\left(-x^c_{2,n}(t)\right) \\ &= \sum_{n=0}^{\infty} \frac{[\lambda(T-t)]^n e^{-\lambda(T-t)}}{n!} \Phi\left(-h^c_{2,n}(t)\right) \end{aligned} \tag{108}$$

where

$$x^c_{1,n}(t) = \frac{\ln \frac{V_t}{B} + [r + \frac{\sigma^2}{2} - \lambda(s-1)](T-t) + n \ln s}{\sqrt{\sigma^2(T-t)}} = -h^c_{1,n}(t),$$

$$x^c_{2,n}(t) = \frac{\ln \frac{V_t}{B} + [r - \frac{\sigma^2}{2} - \lambda(s-1)](T-t) + n \ln s}{\sqrt{\sigma^2(T-t)}} = h^c_{2,n}(t),$$

and the c stand for constant. Given the number of jumps $N_{[t,T]}$ equal n in the time period $[t, T]$, the European call price c_t at time t can be derived as

$$c_t \mid_{N_{[t,T]}=n} = V_t \Phi(x^c_{1,n}(t)) - B e^{-r(T-t)} \Phi(x^c_{2,n}(t)). \tag{109}$$

The unconditional European call price c_t at time t can be written as

$$c_t = \sum_{n=0}^{\infty} \frac{[\lambda(T-t)]^n e^{-\lambda(T-t)}}{n!} [V_t \Phi(x^c_{1,n}(t)) - B e^{-r(T-t)} \Phi(x^c_{2,n}(t))]. \tag{110}$$

The mean of the bond holder's position at current time t is

$$\mathrm{E}^Q_t \left[\frac{B - (B - V_T)_+}{B}\right] \tag{111}$$

$$
\begin{aligned}
&= \frac{B - \mathrm{E}_t^Q[B - V_T]_+}{B} \\
&= \frac{B - p_t e^{r(T-t)}}{B} \\
&= \frac{B - e^{r(T-t)}(c_t + Be^{-r(T-t)} - V_t)}{B} \\
&= \frac{V_t e^{r(T-t)} - c_t e^{r(T-t)}}{B} \\
&= \frac{V_t e^{r(T-t)} - \sum_{n=0}^{\infty} \frac{[\lambda(T-t)]^n e^{-\lambda(T-t)}}{n!}[V_t \Phi(x_{1,n}^c(t)) - Be^{-rT}\Phi(x_{2,n}^c(t))]e^{r(T-t)}}{B} \\
&= \sum_{n=0}^{\infty} \frac{[\lambda(T-t)]^n e^{-\lambda(T-t)}}{n!} \left\{ \frac{V_t e^{r(T-t)}[1 - \Phi(x_{1,n}^c(t))] + B\Phi(x_{2,n}^c(t))}{B} \right\} \\
&= \sum_{n=0}^{\infty} \frac{[\lambda(T-t)]^n e^{-\lambda(T-t)}}{n!} \left\{ \frac{V_t e^{r(T-t)}}{B} \Phi(-x_{1,n}^c(t)) + \Phi(x_{2,n}^c(t)) \right\} \\
&= \sum_{n=0}^{\infty} \frac{[\lambda(T-t)]^n e^{-\lambda(T-t)}}{n!} \left\{ \frac{\Phi(h_{1,n}^c(t))}{d(t)} + \Phi(h_{2,n}^c(t)) \right\}. \qquad (112)
\end{aligned}
$$

The mean recovery rate at time t can be expressed as

$$
\begin{aligned}
\mathrm{E}_t^Q[\delta(T)] &= \frac{\mathrm{E}_t^Q\left[\frac{B-(B-V_T)_+}{B}\right] - 1\,\Pr\{V_T \geq B\}}{\Pr\{V_T < B\}} \\
&= \frac{\sum_{n=0}^{\infty} \frac{[\lambda(T-t)]^n e^{-\lambda(T-t)}}{n!} \left\{ \frac{\Phi(h_{1,n}^c(t))}{d(t)} + \Phi(h_{2,n}^c(t)) \right\} - \sum_{n=0}^{\infty} \frac{(\lambda(T-t))^n e^{-\lambda(T-t)}}{n!} \Phi(h_{2,n}^c(t))}{\sum_{n=0}^{\infty} \frac{(\lambda(T-t))^n e^{-\lambda(T-t)}}{n!} \Phi(-h_{2,n}^c(t))} \\
&= \frac{\sum_{n=0}^{\infty} \frac{(\lambda(T-t))^n e^{-\lambda(T-t)}}{n!} \Phi(h_{1,n}^c(t))}{d(t) \sum_{n=0}^{\infty} \frac{(\lambda(T-t))^n e^{-\lambda(T-t)}}{n!} \Phi(-h_{2,n}^c(t))} \qquad (113) \\
&= \frac{V_t e^{r(T-t)} \sum_{n=0}^{\infty} \frac{(\lambda(T-t))^n e^{-\lambda(T-t)}}{n!} \Phi(-x_{1,n}^c(t))}{B \sum_{n=0}^{\infty} \frac{(\lambda(T-t))^n e^{-\lambda(T-t)}}{n!} \Phi(-x_{2,n}^c(t))}. \qquad (114)
\end{aligned}
$$

From a reduced-form model under risk-neutral valuation in a RTV scheme, the equation (55)

$$
V(t,T) = \mathrm{E}_t^Q \left[\frac{B(t)}{B(T)} (\delta(T) 1_{(\tau \leq T)} + 1_{(\tau > T)}) \right]
$$

can be written as equation (61) as

$$
\begin{aligned}
R(t,T) - r &= -\frac{1}{T-t} \ln \left\{ 1 - \left[1 - \mathrm{E}_t^Q(\delta(T)) \right] \Pr(\tau \leq T) \right\} \\
&= -\frac{1}{T-t} \ln \left\{ 1 - \left[1 - \frac{\sum_{n=0}^{\infty} \frac{(\lambda(T-t))^n e^{-\lambda(T-t)}}{n!} \Phi(h_{1,n}^c(t))}{d(t) \sum_{n=0}^{\infty} \frac{(\lambda(T-t))^n e^{-\lambda(T-t)}}{n!} \Phi(-h_{2,n}^c(t))} \right] \right. \\
&\qquad \left. \times \sum_{n=0}^{\infty} \frac{(\lambda(T-t))^n e^{-\lambda(T-t)}}{n!} \Phi\left(-h_{2,n}^c(t)\right) \right\} \\
&= -\frac{1}{T-t} \ln \left\{ 1 - \left[\sum_{n=0}^{\infty} \frac{(\lambda(T-t))^n e^{-\lambda(t-t)}}{n!} \left\{ \Phi\left(-h_{2,n}^c(t)\right) - \frac{\Phi(h_{1,n}^c(t))}{d(t)} \right\} \right] \right\}
\end{aligned}
$$

$$= \quad -\frac{1}{T-t}\ln\left\{\sum_{n=0}^{\infty}\frac{(\lambda(T-t))^n e^{-\lambda(T-t)}}{n!}\left[\frac{\Phi(h_{1,n}^c(t))}{d(t)}+\Phi\left(h_{2,n}^c(t)\right)\right]\right\}. \tag{115}$$

The credit spread is derived from the yield spread formula for a reduced-form model. This equation shows the yield spread for the corresponding value of a firm process in a reduced-form model. When there is no jump or equivalently let $s = 0$ or $\lambda = 0$, the (115) reduces to a Merton's (1974) credit spread formula equation (4) that

$$R(t,T) - r = -\frac{1}{T-t}\ln\left\{\Phi[h_2(t)] + \frac{1}{d}\Phi[h_1(t)]\right\}$$

which is solved by using boundary conditions on the differential-integral equation. Although the probability of default and mean recovery rate both have explicit forms, we still have difficulty calculating them because we do not know the jump frequency and the jump size. We will address this in the next section. As an alternative to (115), we can write it in the form of bond price as

$$\frac{V(t,T)}{P(t,T)} = \sum_{n=0}^{\infty}\frac{(\lambda(T-t))^n e^{-\lambda(T-t)}}{n!}\left\{\frac{\Phi(h_{1,n}^c(t))}{d(t)}+\Phi\left(h_{2,n}^c(t)\right)\right\}. \tag{116}$$

4.2 Finding the jump size using two seniority classes-constant jumps

We start with the case where a corporate issues bonds of two seniority classes with maturity both at time T. The higher seniority class (class 1) represents p_1 proportion of the total debt and the lower one represents $(1-p_1)$ proportion of the debt. The recovery rate for the class 1 given the bond default is $\delta_1(T) = \min\left(\frac{\delta(T)}{p_1}, 1\right)$. Let $\delta_1^*(T)$ represent the recovery rate only when the recovery is less than 1. The recovery rate for class 2, $\delta_2(T)$, is $\frac{(\delta(T)-p_1)_+}{1-p_1}$. The lower seniority class (class 2) will default if $V_T < B$ and the higher seniority class will default only when $\frac{V_T}{B} < p_1$. The bond price with higher seniority is

$$\begin{aligned}
V_1(t,T) &= \mathrm{E}_t^Q\left[\frac{B(t)}{B(T)}\left\{\delta_1(T)1_{(V_T<B)} + 1_{(V_T\geq B)}\right\}\right] && (117)\\
&= \mathrm{E}_t^Q\left[\frac{B(t)}{B(T)}\left\{\delta_1^*(T)1_{(V_T<Bp_1)} + 1_{(Bp_1\leq V_T<B)} + 1_{(V_T\geq B)}\right\}\right] && (118)\\
&= \mathrm{E}_t^Q\left[\frac{B(t)}{B(T)}\left\{\delta_1^*(T)1_{(V_T<Bp_1)} + 1_{(V_T\geq Bp_1)}\right\}\right]. && (119)
\end{aligned}$$

The bond price for the lower seniority class can be expressed as

$$\begin{aligned}
V_2(t,T) &= \mathrm{E}_t^Q\left[\frac{B(t)}{B(T)}\left\{\delta_2(T)1_{(V_T<B)} + 1_{(V_T\geq B)}\right\}\right] && (120)\\
&= \mathrm{E}_t^Q\left[\frac{B(t)}{B(T)}\left\{\delta_2^*(T)1_{(V_T<B)} + 1_{(V_T\geq B)}\right\}\right]. && (121)
\end{aligned}$$

If the probability of default is independent of recovery rate, then the bond price with higher seniority is

$$\begin{aligned}
V_1(t,T) &= P(t,T)\left\{1-\left(1-\mathrm{E}_t^Q[\delta_1(T)]\right)\Pr\{V_T<B\}\right\} && (122)\\
&= P(t,T)\left\{1-\left(1-\mathrm{E}_t^Q[\delta_1^*(T)]\right)\Pr\{V_T<Bp_1\}\right\}. && (123)
\end{aligned}$$

The bond price with lower seniority is

$$V_2(t,T) \quad = \quad P(t,T)\left\{1-\left(1-\mathrm{E}_t^Q[\delta_2(T)]\right)\Pr\{V_T<B\}\right\}. \tag{124}$$

The credit spread of the bond for the higher seniority class can be expressed as

$$R_1(t,T)-r \quad = \quad -\frac{1}{T-t}\ln\left\{1-\left(1-\mathrm{E}_t^Q[\delta_1(T)]\right)\Pr\{V_T<B\}\right\} \tag{125}$$

$$\quad = \quad -\frac{1}{T-t}\ln\left\{1-\left(1-\mathrm{E}_t^Q[\delta_1^*(T)]\right)\Pr\{V_T<Bp_1\}\right\}. \tag{126}$$

The credit spread for the lower seniority can be expressed as

$$R_2(t,T)-r \quad = \quad -\frac{1}{T-t}\ln\left\{1-\left(1-\mathrm{E}_t^Q[\delta_2(T)]\right)\Pr\{V_T<B\}\right\}. \tag{127}$$

After rearrangement of equation(122) and equation(124), we have

$$\left\{1-\mathrm{E}_t^Q[\delta_i(T)]\right\}\Pr\{V_T<B\} = \frac{P(t,T)-V_i(t,T)}{P(t,T)} \qquad i=1,2. \tag{128}$$

We have the relationship between the mean recovery rate of two different seniority classes in term of bond's price as

$$\frac{1-\mathrm{E}_t^Q[\delta_1(T)]}{1-\mathrm{E}_t^Q[\delta_2(T)]} = \frac{P(t,T)-V_1(t,T)}{P(t,T)-V_2(t,T)}. \tag{129}$$

The recovery rate $\delta_1^*(T)$ is the recovery rate excluding the possibility of $Bp_1 \le V_T < B$. It can be expressed as

$$\delta_1^*(T) \quad = \quad \frac{\delta(T)}{p_1}\ |_{V_T<Bp_1} \tag{130}$$

$$\quad = \quad \frac{V_T}{Bp_1}\ |_{V_T<Bp_1}. \tag{131}$$

We can consider $\delta_1^*(T)$ as the recovery of another bond with a liability of Bp_1. Thus the mean recovery rate $\delta_1^*(T)$ in Q-measure can be calculated from equation(113) and equation(114) as

$$\mathrm{E}_t^Q[\delta_1^*(T)] \quad = \quad \frac{\sum_{n=0}^{\infty}\frac{(\lambda(T-t))^n e^{-\lambda(T-t)}}{n!}\Phi(h_{1,n}^{c,*}(t))}{d^*(t)\sum_{n=0}^{\infty}\frac{(\lambda(T-t))^n e^{-\lambda(T-t)}}{n!}\Phi(-h_{2,n}^{c,*}(t))} \tag{132}$$

$$\quad = \quad \frac{V_t e^{r(T-t)}\sum_{n=0}^{\infty}\frac{(\lambda(T-t))^n e^{-\lambda(T-t)}}{n!}\Phi(-x_{1,n}^{c,*}(t))}{Bp_1\sum_{n=0}^{\infty}\frac{(\lambda(T-t))^n e^{-\lambda(T-t)}}{n!}\Phi(-x_{2,n}^{c,*}(t))} \tag{133}$$

where

$$d^*(t) = \frac{Bp_1}{V_t e^{r(T-t)}} = p_1 d(t), \tag{134}$$

$$\begin{aligned} x_{1,n}^{c,*}(t) &= \frac{\ln\frac{V_t}{Bp_1}+[r+\frac{\sigma^2}{2}-\lambda(s-1)](T-t)+n\ln s}{\sqrt{\sigma^2(T-t)}} \\ &= \frac{\ln\frac{V_t}{B}+([r+\frac{\sigma^2}{2}-\lambda(s-1)](T-t)+n\in s-\ln p_1}{\sqrt{\sigma^2(T-t)}} \\ &= x_{1,n}^{c}(t)-\frac{\ln p_1}{\sqrt{\sigma^2(T-t)}} \\ &= -h_{1,n}^{c,*}(t), \end{aligned}$$

and

$$
\begin{aligned}
x_{2,n}^{c,*}(t) &= \frac{\ln \frac{V_t}{Bp_1} + [r - \frac{\sigma^2}{2} - \lambda(s-1)](T-t) - n\ln s}{\sqrt{\sigma^2(T-t)}} \\
&= \frac{\ln \frac{V_t}{B} + [r - \frac{\sigma^2}{2} - \lambda(s-1)](T-t) - n\ln s - \ln p_1}{\sqrt{\sigma^2(T-t)}} \\
&= x_{2,n}^{c}(t) - \frac{\ln p_1}{\sqrt{\sigma^2(T-t)}} \\
&= h_{2,n}^{c,*}(t).
\end{aligned}
$$

The probability that $V_T < Bp_1$ is

$$
\begin{aligned}
\Pr\{V_T < Bp_1\} &= \sum_{n=0}^{\infty} \frac{(\lambda(T-t))^n e^{-\lambda(T-t)}}{n!} \Phi\left(-x_{2,n}^{c,*}(t)\right) \\
&= \sum_{n=0}^{\infty} \frac{(\lambda(T-t))^n e^{-\lambda(T-t)}}{n!} \Phi\left(-h_{2,n}^{c,*}(t)\right). \qquad (135)
\end{aligned}
$$

The mean recovery rate for the higher seniority can be derived as

$$
\begin{aligned}
&\mathrm{E}_t^Q[\delta_1(T)] \\
&= \frac{\mathrm{E}_t^Q[\delta_1^*(T)]\Pr\{V_T < Bp_1\} + 1\Pr\{Bp_1 \le V_T < B\}}{\Pr\{V_T < B\}} \\
&= \frac{\frac{\sum_{n=0}^{\infty} \frac{(\lambda(T-t))^n e^{-\lambda(T-t)}}{n!} \Phi(h_{1,n}^{c,*}(t))}{d^*(t)\sum_{n=0}^{\infty} \frac{(\lambda(T-t))^n e^{-\lambda(T-t)}}{n!} \Phi(-h_{2,n}^{c,*}(t))} \sum_{n=0}^{\infty} \frac{(\lambda(T-t))^n e^{-\lambda(T-t)}}{n!} \Phi(-h_{2,n}^{c,*}(t))}{\sum_{n=0}^{\infty} \frac{(\lambda(T-t))^n e^{-\lambda(T-t)}}{n!} \Phi\left(-h_{2,n}(t)\right)} \\
&\quad + \frac{\left[\sum_{n=0}^{\infty} \frac{(\lambda(T-t))^n e^{-\lambda(T-t)}}{n!} \Phi\left(-h_{2,n}(t)\right) - \sum_{n=0}^{\infty} \frac{(\lambda(T-t))^n e^{-\lambda(T-t)}}{n!} \Phi\left(-h_{2,n}^{c,*}(t)\right)\right]}{\sum_{n=0}^{\infty} \frac{(\lambda T)^n e^{-\lambda T}}{n!} \Phi\left(-h_{2,n}(t)\right)} \\
&= 1 - \frac{\sum_{n=0}^{\infty} \frac{(\lambda(T-t))^n e^{-\lambda(T-t)}}{n!} \left[\Phi\left(-h_{2,n}^{c,*}(t)\right) - \frac{\Phi(h_{1,n}^{c,*}(t))}{d^*(t)}\right]}{\sum_{n=0}^{\infty} \frac{(\lambda(T-t))^n e^{-\lambda(T-t)}}{n!} \Phi\left(-h_{2,n}(t)\right)}. \qquad (136)
\end{aligned}
$$

Now consider

$$
p_1\left[\frac{\delta(T)}{p_1} \wedge 1\right] + (1-p_1)\frac{(\delta(T)-p_1)_+}{1-p_1} = [\delta(T) \wedge p_1] + (\delta(T)-p_1)_+ = \delta(T). \qquad (137)
$$

Taking the expected valued at time t, we have

$$
p_1\mathrm{E}_t^Q\left[\frac{\delta(T)}{p_1} \wedge 1\right] + (1-p_1)\mathrm{E}_t^Q\left[\frac{(\delta(T)-p_1)_+}{1-p_1}\right] = \mathrm{E}_t^Q[\delta(T)] \qquad (138)
$$

and

$$
p_1\mathrm{E}_t^Q\left[\delta_1(T)\right] + (1-p_1)\mathrm{E}_t^Q\left[\delta_2(T)\right] = \mathrm{E}_t^Q[\delta(T)]. \qquad (139)
$$

Thus, the mean recovery rate for the lower seniority can be derived as

$$\begin{aligned}
&\mathrm{E}_t^Q[\delta_2(T)] \\
&= \frac{\mathrm{E}_t^Q[\delta(T)] - p_1 \mathrm{E}_t^Q[\delta_1(T)]}{1-p_1} \qquad (140) \\
&= \frac{\frac{\sum_{n=0}^{\infty} \frac{(\lambda(T-t))^n e^{-\lambda(T-t)}}{n!} \Phi(h_{1,n}(t))}{d(t)\sum_{n=0}^{\infty} \frac{(\lambda(T-t))^n e^{-\lambda(T-t)}}{n!} \Phi(-h_{2,n}(t))} - p_1 + p_1 \frac{\sum_{n=0}^{\infty} \frac{(\lambda(T-t))^n e^{-\lambda(T-t)}}{n!} \left[\Phi\left(-h_{2,n}^{c,*}(t)\right) - \frac{\Phi(h_{1,n}^{c,*}(t))}{d^*(t)}\right]}{\sum_{n=0}^{\infty} \frac{(\lambda(T-t))^n e^{-\lambda(T-t)}}{n!} \Phi(-h_{2,n}(t))}}{1-p_1} \\
&= \frac{\frac{\sum_{n=0}^{\infty} \frac{(\lambda(T-t))^n e^{-\lambda(T-t)}}{n!} [\Phi(h_{1,n}(t)) - \Phi(h_{1,n}^{c,*}(t))]}{d(t)\sum_{n=0}^{\infty} \frac{(\lambda(T-t))^n e^{-\lambda(T-t)}}{n!} \Phi(-h_{2,n}(t))} - p_1 + 1 - 1 + p_1 \frac{\sum_{n=0}^{\infty} \frac{(\lambda(T-t))^n e^{-\lambda(T-t)}}{n!} \Phi\left(-h_{2,n}^{c,*}(t)\right)}{\sum_{n=0}^{\infty} \frac{(\lambda(T-t))^n e^{-\lambda(T-t)}}{n!} \Phi(-h_{2,n}(t))}}{1-p_1} \\
&= 1 - \frac{\sum_{n=0}^{\infty} \frac{(\lambda(T-t))^n e^{-\lambda(T-t)}}{n!} \left\{\left[\Phi\left(-h_{2,n}(t)\right) - p_1 \Phi\left(-h_{2,n}(t)\right)\right] - \frac{\Phi(h_{1,n}(t)) - \Phi(h_{1,n}^{c,*}(t))}{d(t)}\right\}}{(1-p_1)\sum_{n=0}^{\infty} \frac{(\lambda(T-t))^n e^{-\lambda(T-t)}}{n!} \Phi\left(-h_{2,n}(t)\right)}. \qquad (141)
\end{aligned}$$

Substituting $\mathrm{E}_t^Q[\delta_1(T)]$ and $\mathrm{E}_t^Q[\delta_2(T)]$ into equation (129), we have

$$\frac{\frac{\sum_{n=0}^{\infty} \frac{(\lambda(T-t))^n e^{-\lambda(T-t)}}{n!} \left[\Phi\left(-h_{2,n}^{c,*}(t)\right) - \frac{\Phi(h_{1,n}^{c,*}(t))}{d^*(t)}\right]}{\sum_{n=0}^{\infty} \frac{(\lambda(T-t))^n e^{-\lambda(T-t)}}{n!} \Phi(-h_{2,n}(t))}}{\frac{\sum_{n=0}^{\infty} \frac{(\lambda(T-t))^n e^{-\lambda(T-t)}}{n!} \left\{[\Phi(-h_{2,n}(t)) - p_1 \Phi(-h_{2,n}(t))] - \frac{\Phi(h_{1,n}(t)) - \Phi(h_{1,n}^{c,*}(t))}{d(t)}\right\}}{(1-p_1)\sum_{n=0}^{\infty} \frac{(\lambda(T-t))^n e^{-\lambda(T-t)}}{n!} \Phi(-h_{2,n}(t))}} = \frac{P(t,T) - V_1(t,T)}{P(t,T) - V_2(t,T)}. \qquad (142)$$

After rearrangement, we have

$$\begin{aligned}
&\frac{\sum_{n=0}^{\infty} \frac{(\lambda(T-t))^n e^{-\lambda(T-t)}}{n!} \left[p_1 \Phi(-h_{2,n}^{c,*}(t)) - \frac{\Phi(h_{1,n}^{c,*}(t))}{d(t)}\right]}{\sum_{n=0}^{\infty} \frac{(\lambda(T-t))^n e^{-\lambda(T-t)}}{n!} \left\{\left[\Phi\left(-h_{2,n}(t)\right) - p_1 \Phi(-h_{2,n}^{c,*}(t))\right] - \frac{[\Phi(h_{1,n}(t)) - \Phi(h_{1,n}^{c,*}(t))]}{d(t)}\right\}} \\
&= \frac{p_1[P(t,T) - V_1(t,T)]}{(1-p_1)[P(t,T) - V_2(t,T)]}. \qquad (143)
\end{aligned}$$

There are two unknown parameters s and λ in equation (143). With only one constraint, the solution cannot be derived. With one more constraint (116)

$$\frac{V(t,T)}{P(t,T)} = \sum_{n=0}^{\infty} \frac{(\lambda(T-t))^n e^{-\lambda(T-t)}}{n!} \left\{\frac{\Phi(h_{1,n}^c(t))}{d(t)} + \Phi\left(h_{2,n}^c(t)\right)\right\}$$

where $V(t,T) = p_1 V_1(t,T) + (1-p_1)V_2(t,T)$, we can obtain the s and λ numerically. This means that the price difference between the bond issued by one firm with different seniorities reflect what the investors expect about the jump behavior in Q-measure.

If the jump size and jump frequency can be derived uniquely, thus the default probability and mean recovery rate are unique. If there are infinite combinations of jump size and jump frequency, the Q-measure is not unique. In the case where a firm issue bonds of more than two seniorities, the jump size and jump frequency calculated using the first two seniorities should be consistent with those calculated using other seniorities. If not, the no arbitrage assumption is violated due to the investors insufficient knowledge about the jump size and jump frequency. The arbitrage opportunity will vanish if the arbitragers know this opportunity and trade on this opportunity.

In the case where the two bonds issued have different maturity T_1 and T_2, then equation (129) is revised as

$$\frac{1-\mathrm{E}_t^Q[\delta_1(T_1)]\Pr\{V_{T_1}<B\}}{1-\mathrm{E}_t^Q[\delta_2(T_2)]\Pr\{V_{T_2}<B\}}=\frac{P(t,T_2)[P(t,T_1)-V_1(t,T_1)]}{P(t,T_1)[P(t,T_2)-V_2(t,T_2)]}. \tag{144}$$

The first constraint equation (143) is revised as

$$\frac{\sum_{n=0}^{\infty}\frac{(\lambda(T-t))^n e^{-\lambda(T-t)}\left[p_1\Phi(-h_{2,n}^{c,*}(t))-\frac{\Phi(h_{1,n}^{c,*}(t))}{d(t)}\right]}{n!}}{\sum_{n=0}^{\infty}\frac{(\lambda(T-t))^n e^{-\lambda(T-t)}\left\{[\Phi(-h_{2,n}(t))-p_1\Phi(-h_{2,n}^{c,*}(t))]-\frac{[\Phi(h_{1,n}(t))-\Phi(h_{1,n}^{c,*}(t))]}{d(t)}\right\}}{n!}}$$
$$\times\frac{\sum_{n=0}^{\infty}\frac{[\lambda(T-t)]^n e^{-\lambda(T-t)}\Phi\left(-h_{2,n}^{c}(t,T_1)\right)}{n!}}{\sum_{n=0}^{\infty}\frac{[\lambda(T-t)]^n e^{-\lambda(T-t)}\Phi\left(-h_{2,n}^{c}(t,T_2)\right)}{n!}}=\frac{p_1P(t,T_2)[P(t,T_1)-V_1(t,T_1)]}{(1-p_1)P(t,T_1)[P(t,T_2)-V_2(t,T_2)]} \tag{145}$$

where

$$h_{2,n}^c(t,T)=\frac{\ln\frac{V_t}{B}+[r-\frac{\sigma^2}{2}-\lambda(s-1)](T-t)+n\ln s}{\sqrt{\sigma^2(T-t)}}.$$

The second constraint is given by

$$\sum_{n=0}^{\infty}\frac{(\lambda(T-t))^n e^{-\lambda(T-t)}}{n!}\left[\Phi\left(-h_{2,n}^{c,*}(t)\right)-\frac{\Phi(h_{1,n}^{c,*}(t))}{d^*(t)}\right]=\frac{P(t,T_1)-V_1(t,T_1)}{P(t,T_1)}, \tag{146}$$

or

$$\sum_{n=0}^{\infty}\frac{(\lambda(T-t))^n e^{-\lambda(T-t)}}{n!}\frac{\left\{[\Phi\left(-h_{2,n}(t)\right)-p_1\Phi\left(-h_{2,n}(t)\right)]-\frac{\Phi(h_{1,n}(t))-\Phi(h_{1,n}^{c,*}(t))}{d(t)}\right\}}{(1-p_1)}$$
$$=\frac{P(t,T_2)-V_2(t,T_2)}{P(t,T_2)}. \tag{147}$$

4.3 Unifying structural jump-diffusion model with reduced-form models-lognormal jumps

We consider a case that the impulse function Y or the jump amplitude plus one $J+1$ is lognormal. Let

$$\ln Y_i \sim N(\mu_1,\sigma_1) \qquad i\in\{0,1,2...\} \tag{148}$$

with probability density function as

$$f(y) = \frac{1}{\sqrt{(2\pi)}\sigma_1 y} e^{-\frac{(\ln y - \mu_1)^2}{2\sigma_1^2}} \tag{149}$$

where $\mu_y = \mu_J + 1 = e^{\mu_1 + \frac{\sigma_1^2}{2}}$ and $\sigma_Y = \sigma_J = e^{2(\mu_1+\sigma_1^2)} - e^{2\mu_1+\sigma_1^2}$. From equation (102), the value of a firm follows

$$\ln V_t = \ln V_0 + \left[r - \frac{\sigma^2}{2} - \lambda(\mu_Y - 1)\right] t + \sigma Z_t + \sum_{i=1}^{N_t} \ln Y_i.$$

Conditioning on the number of jump, the value of a firm follows

$$\ln V_t \mid_{N_t=n} \sim N\left\{\ln V_0 + \left[r - \frac{\sigma^2}{2} - \lambda(\mu_Y - 1)\right] t + n\mu_1, \sqrt{\sigma^2 t + n\sigma_1^2}\right\}. \tag{150}$$

At the current time t, the probability of default at bond's maturity is derived similar to Zhou (1997) as

$$\begin{aligned} \Pr(\tau = T) &= \sum_{n=0}^{\infty} \frac{(\lambda(T-t))^n e^{-\lambda(T-t)}}{n!} \Phi\left(-\frac{\ln\frac{V_t}{B} + [r - \frac{\sigma^2}{2} - \lambda(\mu_Y - 1)](T-t) + n\mu_1}{\sqrt{\sigma^2(T-t) + n\sigma_1^2}}\right) \\ &= \sum_{n=0}^{\infty} \frac{(\lambda(T-t))^n e^{-\lambda(T-t)}}{n!} \Phi\left(-x_{2,n}(t)\right) &(151) \\ &= \sum_{n=0}^{\infty} \frac{(\lambda(T-t))^n e^{-\lambda(T-t)}}{n!} \Phi\left(-h_{2,n}(t)\right) &(152) \end{aligned}$$

where

$$x_{1,n}(t) = \frac{\ln\frac{V_t}{B} + [r + \frac{\sigma^2}{2} - \lambda(\mu_Y - 1)](T-t) + n\mu_1}{\sqrt{\sigma^2(T-t) + n\sigma_1^2}} = -h_{1,n}(t),$$

and

$$x_{2,n}(t) = \frac{\ln\frac{V_t}{B} + [r - \frac{\sigma^2}{2} - \lambda(\mu_Y - 1)](T-t) + n\mu_1}{\sqrt{\sigma^2(T-t) + n\sigma_1^2}} = h_{2,n}(t).$$

At the current time t, given the number of jumps $N_{[t,T]}$ in the time period $[t, T]$, the European call price can be derived as

$$c_t \mid_{N_t=n} = V_t \Phi(x_{1,n}(t)) - Be^{-r(T-t)}\Phi(x_{2,n}(t)). \tag{153}$$

The unconditional European call price at time T is

$$c_t = \sum_{n=0}^{\infty} \frac{(\lambda(T-t))^n e^{-\lambda(T-t)}}{n!} [V_t \Phi(x_{1,n}(t)) - Be^{-r(T-t)}\Phi(x_{2,n}(t))]. \tag{154}$$

The mean of the bond holder's position at current time t is

$$\mathrm{E}_t^Q \left[\frac{B - (B - V_T)_+}{B}\right] \tag{155}$$

$$
\begin{aligned}
&= \frac{B - \mathrm{E}_t^Q[B - V_T]_+}{B} \\
&= \frac{B - p_t e^{r(T-t)}}{B} \\
&= \frac{B - e^{r(T-t)}(c_t + Be^{-r(T-t)} - V_t)}{B} \\
&= \frac{V_t e^{r(T-t)} - c_t e^{r(T-t)}}{B} \\
&= \frac{V_t e^{r(T-t)} - \sum_{n=0}^{\infty} \frac{(\lambda(T-t))^n e^{-\lambda(T-t)}}{n!}[V_t \Phi(x_{1,n}(t)) - Be^{-r(T-t)}\Phi(x_{2,n}(t))]e^{r(T-t)}}{B} \\
&= \sum_{n=0}^{\infty} \frac{(\lambda(T-t))^n e^{-\lambda(T-t)}}{n!} \left\{ \frac{V_t e^{r(T-t)}[1 - \Phi(x_{1,n}(t))] + B\Phi(x_{2,n}(t))}{B} \right\} \\
&= \sum_{n=0}^{\infty} \frac{(\lambda(T-t))^n e^{-\lambda(T-t)}}{n!} \left\{ \frac{V_t e^{rT}}{B} \Phi(-x_{1,n}(t)) + \Phi(x_{2,n}(t)) \right\} \\
&= \sum_{n=0}^{\infty} \frac{(\lambda(T-t))^n e^{-\lambda(T-t)}}{n!} \left\{ \frac{\Phi(h_{1,n}(t))}{d(t)} + \Phi(h_{2,n}(t)) \right\}. \qquad (156)
\end{aligned}
$$

The mean recovery rate can be expressed as

$$
\begin{aligned}
\mathrm{E}_t^Q[\delta(T)] &= \frac{\mathrm{E}_t^Q\left[\frac{B-(B-V_T)_+}{B}\right] - 1\,\Pr\{V_T \geq B\}}{\Pr\{V_T < B\}} \\
&= \frac{\sum_{n=0}^{\infty} \frac{(\lambda(T-t))^n e^{-\lambda(T-t)}}{n!} \left\{ \frac{\Phi(h_{1,n}(t))}{d(t)} + \Phi(h_{2,n}(t)) \right\} - \sum_{n=0}^{\infty} \frac{(\lambda(T-t))^n e^{-\lambda(T-t)}}{n!} \Phi(h_{2,n}(t))}{\sum_{n=0}^{\infty} \frac{(\lambda(T-t))^n e^{-\lambda(T-t)}}{n!} \Phi(-h_{2,n}(t))} \\
&= \frac{\sum_{n=0}^{\infty} \frac{(\lambda(T-t))^n e^{-\lambda(T-t)}}{n!} \Phi(h_{1,n}(t))}{d(t) \sum_{n=0}^{\infty} \frac{(\lambda(T-t))^n e^{-\lambda(T-t)}}{n!} \Phi(-h_{2,n}(t))} \qquad (157) \\
&= \frac{V_t e^{r(T-t)} \sum_{n=0}^{\infty} \frac{(\lambda(T-t))^n e^{-\lambda(T-t)}}{n!} \Phi(-x_{1,n}(t))}{B \sum_{n=0}^{\infty} \frac{(\lambda(T-t))^n e^{-\lambda(T-t)}}{n!} \Phi(-x_{2,n}(t))}. \qquad (158)
\end{aligned}
$$

The yield spread from a reduced-form model based on the RTV scheme is

$$
\begin{aligned}
R(t,T) - r &= -\frac{1}{T-t} \ln\left[1 - \left(1 - \mathrm{E}_t^Q(\delta(T))\right) \Pr(\tau \leq T)\right] \\
&= -\frac{1}{T-t} \ln\left[1 - \left(1 - \frac{\sum_{n=0}^{\infty} \frac{(\lambda T)^n e^{-\lambda T}}{n!} \Phi(h_{1,n}(t))}{d(t) \sum_{n=0}^{\infty} \frac{(\lambda(T-t))^n e^{-\lambda(T-t)}}{n!} \Phi(-h_{2,n}(t))}\right) \right. \\
&\qquad \left. \times \sum_{n=0}^{\infty} \frac{(\lambda(T-t))^n e^{-\lambda(T-t)}}{n!} \Phi\left(-h_{2,n}(t)\right) \right] \\
&= -\frac{1}{T-t} \ln\left[1 - \left(\sum_{n=0}^{\infty} \frac{(\lambda(T-t))^n e^{-\lambda(T-t)}}{n!} \left[\Phi\left(-h_{2,n}(t)\right) - \frac{\Phi(h_{1,n}(t))}{d(t)}\right]\right)\right] \\
&= -\frac{1}{T-t} \ln\left[\sum_{n=0}^{\infty} \frac{(\lambda(T-t))^n e^{-\lambda(T-t)}}{n!} \left\{ \frac{\Phi(h_{1,n}(t))}{d(t)} + \Phi\left(h_{2,n}(t)\right) \right\}\right]. \qquad (159)
\end{aligned}
$$

An explicit form for credit spread is thus derived. When there is no jump or equivalently letting $\lambda = 0$, the equation (159) reduces to a Merton's (1974) credit spread formula equation (4)

$$R(t,T) - r = -\frac{1}{T-t} \ln \left\{ \Phi[h_2(t)] + \frac{1}{d} \Phi[h_1(t)] \right\}.$$

Alternatively, we have

$$\frac{V(t,T)}{P(t,T)} = \sum_{n=0}^{\infty} \frac{(\lambda(T-t))^n e^{-\lambda(T-t)}}{n!} \left\{ \frac{\Phi(h_{1,n}(t))}{d(t)} + \Phi\left(h_{2,n}(t)\right) \right\}. \tag{160}$$

In order to calculate the default probability and mean recovery rate, we need to know the mean and variance of the normal jumps. There are three unknown parameter, λ, μ_Y, and σ_Y, we at least need two more constraints besides equation (160) to derive these parameters. We will introduce it in the next section.

4.4 Finding the jump size using three seniority classes-lognormal jumps

We assume a firm issued bonds consisted of three seniority classes with maturity at time T. The higher seniority class represents p_1 proportion of the total debt, the second lower one represents p_2 proportion of the debt, and the third seniority class represent $(1 - p_1 - p_2)$ proportion of the debt. The recovery rate for the class 1 given the bond default is $\delta_1(T) = \min\left(\frac{\delta(T)}{p_1}, 1\right)$. Let $\delta_1^*(T)$ represent the recovery rate only when the recovery is less than 1. The recovery rate for class 2 is $\delta_2(T) = \min\left(\frac{(\delta(T)-p_1)_+}{p_2}, 1\right)$. Let $\delta_2^*(T)$ represent the recovery rate for the seniority class 2 only when the recovery is less than 1. The recovery rate for the class 3 is $\delta_3(T) = \frac{(\delta(T)-p_1-p_2)_+}{1-p_1-p_2}$. Following the results of section (4.2), the bond price of seniority class 1 is

$$\begin{aligned} V_1(t,T) &= \mathrm{E}_t^Q \left[\frac{B(t)}{B(T)} \left\{ \delta_1(T) 1_{(V_T<B)} + 1_{(V_T \geq B)} \right\} \right] & (161) \\ &= \mathrm{E}_t^Q \left[\frac{B(t)}{B(T)} \left\{ \delta_1^*(T) 1_{(V_T<Bp_1)} + 1_{(V_T \geq Bp_1)} \right\} \right]. & (162) \end{aligned}$$

The bond price of seniority class 2 can be expressed as

$$\begin{aligned} V_2(t,T) &= \mathrm{E}_t^Q \left[\frac{B(t)}{B(T)} \left\{ \delta_2(T) 1_{(V_T<B)} + 1_{(V_T \geq B)} \right\} \right] & (163) \\ &= \mathrm{E}_t^Q \left[\frac{B(t)}{B(T)} \left\{ \delta_2^*(T) 1_{[V_T<B(p_1+p_2)]} + 1_{[V_T \geq B(p_1+p_2)]} \right\} \right]. & (164) \end{aligned}$$

The bond price of seniority class 3 can be expressed as

$$\begin{aligned} V_3(t,T) &= \mathrm{E}_t^Q \left[\frac{B(t)}{B(T)} \left\{ \delta_3(T) 1_{(V_T<B)} + 1_{(V_T \geq B)} \right\} \right] & (165) \\ &= \mathrm{E}_t^Q \left[\frac{B(t)}{B(T)} \left\{ \delta_3^*(T) 1_{(V_T<B)} + 1_{(V_T \geq B)} \right\} \right]. & (166) \end{aligned}$$

If the probability of default is independent of the recovery rate, then

$$\left\{ 1 - \mathrm{E}_t^Q[\delta_i(T)] \right\} \Pr\{V_T < B\} = \frac{P(t,T) - V_i(t,T)}{P(t,T)} \qquad i = 1, 2, 3. \tag{167}$$

We have the relationship between the mean recovery rate of two different seniority classes in term of the bond's price as

$$\frac{1-\mathrm{E}_t^Q[\delta_1(T)]}{1-\mathrm{E}_t^Q[\delta_2(T)]}=\frac{P(t,T)-V_1(t,T)}{P(t,T)-V_2(t,T)} \tag{168}$$

and

$$\frac{1-\mathrm{E}_t^Q[\delta_1(T)]}{1-\mathrm{E}_t^Q[\delta_3(T)]}=\frac{P(t,T)-V_1(t,T)}{P(t,T)-V_3(t,T)}. \tag{169}$$

The recovery rate $\delta_1^*(T)$ is the recovery rate excluding the possibility of $Bp_1 \leq V_T < B$. It can be expressed as

$$\delta_1^*(T) = \frac{\delta(T)}{p_1}\,|_{V_T<Bp_1} \tag{170}$$

$$= \frac{V_T}{Bp_1}\,|_{V_T<Bp_1}\,. \tag{171}$$

We can consider $\delta_1^*(T)$ as the recovery of another bond with a liability of Bp_1. The mean recovery rate $\delta_1^*(T)$ in Q-measure can be derived from equation(157) and equation(158) as

$$\mathrm{E}_t^Q[\delta_1^*(T)] = \frac{\sum_{n=0}^{\infty}\frac{(\lambda(T-t))^n e^{-\lambda(T-t)}}{n!}\Phi(h_{1,n}^*(t))}{d^*(t)\sum_{n=0}^{\infty}\frac{(\lambda(T-t))^n e^{-\lambda(T-t)}}{n!}\Phi(-h_{2,n}^*(t))} \tag{172}$$

$$= \frac{V_t e^{r(T-t)}\sum_{n=0}^{\infty}\frac{(\lambda(T-t))^n e^{-\lambda(T-t)}}{n!}\Phi(-x_{1,n}^*(t))}{Bp_1\sum_{n=0}^{\infty}\frac{(\lambda(T-t))^n e^{-\lambda(T-t)}}{n!}\Phi(-x_{2,n}^*(t))} \tag{173}$$

where

$$d^*(t)=\frac{Bp_1}{V_t e^{r(T-t)}}=p_1 d(t), \tag{174}$$

$$x_{1,n}^*(t) = \frac{\ln\frac{V_t}{Bp_1}+(r+\frac{\sigma^2}{2}-\lambda\mu_J)(T-t)+n\mu_1}{\sqrt{\sigma^2(T-t)+n\sigma_1^2}} \tag{175}$$

$$= \frac{\ln\frac{V_t}{B}+(r+\frac{\sigma^2}{2}-\lambda\mu_Y-\lambda)(T-t)+n\mu_1-\ln(p_1)}{\sqrt{\sigma^2(T-t)+n\sigma_1^2}} \tag{176}$$

$$= x_{1,n}(t)-\frac{\ln(p_1)}{\sqrt{\sigma^2(T-t)+n\sigma_1^2}} \tag{177}$$

$$= -h_{1,n}^*(t), \tag{178}$$

and

$$x_{2,n}^*(t) = \frac{\ln\frac{V_t}{Bp_1}+(r-\frac{\sigma^2}{2}-\lambda\mu_J)(T-t)+n\mu_1}{\sqrt{\sigma^2(T-t)+n\sigma_1^2}} \tag{179}$$

$$= \frac{\ln\frac{V_t}{B}+(r-\frac{\sigma^2}{2}-\lambda\mu_Y-\lambda)(T-t)+n\mu_1-\ln(p_1)}{\sqrt{\sigma^2(T-t)+n\sigma_1^2}} \tag{180}$$

$$= x_{2,n}(t)-\frac{\ln(p_1)}{\sqrt{\sigma^2(T-t)+n\sigma_1^2}} \tag{181}$$

$$= h_{2,n}^*(t). \tag{182}$$

The probability that $V_T < Bp_1$ is

$$\Pr\{V_T < Bp_1\} = \sum_{n=0}^{\infty} \frac{(\lambda(T-t))^n e^{-\lambda(T-t)}}{n!} \Phi\left(-x_{2,n}^*(t)\right) \tag{183}$$

$$= \sum_{n=0}^{\infty} \frac{(\lambda(T-t))^n e^{-\lambda(T-t)}}{n!} \Phi\left(-h_{2,n}^*(t)\right). \tag{184}$$

The mean recovery rate for the higher seniority can be derived as

$$\mathrm{E}_t^Q[\delta_1(T)]$$

$$= \frac{\mathrm{E}_t^Q[\delta_1^*(T)] \Pr\{V_T < Bp_1\} + 1 \Pr\{Bp_1 \le V_T < B\}}{\Pr\{V_T < B\}} \tag{185}$$

$$= \frac{\frac{\sum_{n=0}^{\infty} \frac{(\lambda(T-t))^n e^{-\lambda(T-t)}}{n!} \Phi(h_{1,n}^*(t))}{d^*(t) \sum_{n=0}^{\infty} \frac{(\lambda(T-t))^n e^{-\lambda(T-t)}}{n!} \Phi(-h_{2,n}^*(t))} \sum_{n=0}^{\infty} \frac{(\lambda(T-t))^n e^{-\lambda(T-t)}}{n!} \Phi(-h_{2,n}^*(t))}{\sum_{n=0}^{\infty} \frac{(\lambda(T-t))^n e^{-\lambda(T-t)}}{n!} \Phi\left(-h_{2,n}(t)\right)}$$

$$+ \frac{\left[\sum_{n=0}^{\infty} \frac{(\lambda(T-t))^n e^{-\lambda(T-t)}}{n!} \Phi\left(-h_{2,n}(t)\right) - \sum_{n=0}^{\infty} \frac{(\lambda(T-t))^n e^{-\lambda(T-t)}}{n!} \Phi\left(-h_{2,n}^*(t)\right)\right]}{\sum_{n=0}^{\infty} \frac{(\lambda(T-t))^n e^{-\lambda(T-t)}}{n!} \Phi\left(-h_{2,n}(t)\right)}$$

$$= \frac{\sum_{n=0}^{\infty} \frac{(\lambda(T-t))^n e^{-\lambda(T-t)}}{n!} \Phi(h_{1,n}^*(t))}{d^*(t) \sum_{n=0}^{\infty} \frac{(\lambda(T-t))^n e^{-\lambda(T-t)}}{n!} \Phi(-h_{2,n}(t))} + 1 - \frac{\sum_{n=0}^{\infty} \frac{(\lambda(T-t))^n e^{-\lambda(T-t)}}{n!} \Phi\left(-h_{2,n}^*(t)\right)}{\sum_{n=0}^{\infty} \frac{(\lambda(T-t))^n e^{-\lambda(T-t)}}{n!} \Phi\left(-h_{2,n}(t)\right)}$$

$$= 1 - \frac{\sum_{n=0}^{\infty} \frac{(\lambda(T-t))^n e^{-\lambda(T-t)}}{n!} \left[\Phi\left(-h_{2,n}^*(t)\right) - \frac{\Phi(h_{1,n}^*(t))}{d(t) p_1}\right]}{\sum_{n=0}^{\infty} \frac{(\lambda(T-t))^n e^{-\lambda(T-t)}}{n!} \Phi\left(-h_{2,n}(t)\right)}. \tag{186}$$

Now consider

$$p_1 \left[\frac{\delta(T)}{p_1} \wedge 1\right] + p_2 \left[\frac{(\delta(T) - p_1)_+}{p_2} \wedge 1\right] + (1 - p_1 - p_2) \frac{(\delta(T) - p_1 - p_2)_+}{1 - p_1 - p_2} = \delta(T). \tag{187}$$

Taking the expected valued at time t, we have

$$p_1 \mathrm{E}_t^Q \left[\frac{\delta(T)}{p_1} \wedge 1\right] + p_2 \mathrm{E}_t^Q \left[\frac{(\delta(T) - p_1)_+}{p_2} \wedge 1\right] + (1 - p_1 - p_2) \mathrm{E}_t^Q \left[\frac{(\delta(T) - p_1 - p_2)_+}{1 - p_1 - p_2}\right]$$
$$= \mathrm{E}_t^Q[\delta(T)], \tag{188}$$

and

$$p_1 \mathrm{E}_t^Q \left[\delta_1(T)\right] + p_2 \mathrm{E}_t^Q \left[\delta_2(T)\right] + (1 - p_1 - p_2) \mathrm{E}_t^Q \left[\delta_3(T)\right] = \mathrm{E}_t^Q[\delta(T)]. \tag{189}$$

The mean recovery rate for seniority class 3, $\mathrm{E}_t^Q\left[\delta_3(T)\right] = \mathrm{E}_t^Q\left[\frac{(\delta(T)-p_1-p_2)_+}{1-p_1-p_2}\right]$, can be considered as that in a two seniority classes case but the first seniority represent $(p_1 + p_2)$ proportion of the debt. Follow equation (141), we have

$$\mathrm{E}_t^Q[\delta_3(T)]$$

$$= 1 - \frac{\sum_{n=0}^{\infty} \frac{(\lambda(T-t))^n e^{-\lambda(T-t)}}{n!} \left[\frac{\Phi(h_{1,n}^{**}(t)) - \Phi(h_{1,n}(t))}{d(t)} + \Phi(-h_{2,n}(t)) - (p_1 + p_2) \Phi\left(-h_{2,n}^{**}(t)\right)\right]}{(1 - p_1 - p_2) \sum_{n=0}^{\infty} \frac{(\lambda(T-t))^n e^{-\lambda(T-t)}}{n!} \Phi(-h_{2,n}(t))} \tag{190}$$

where

$$x_{1,n}^{**} = \frac{\ln \frac{V_t}{B(p_1+p_2)} + [r + \frac{\sigma^2}{2} + \lambda(\mu_Y - 1)](T-t) - n\mu_Y}{\sqrt{\sigma^2(T-t) + n\sigma_Y^2}} = -h_{1,n}^{**}(t),$$

and

$$x_{2,n}^{**} = \frac{\ln \frac{V_t}{B(p_1+p_2)} + [r - \frac{\sigma^2}{2} + \lambda(\mu_Y - 1)](T-t) - n\mu_Y}{\sqrt{\sigma^2(T-t) + n\sigma_Y^2}} = h_{2,n}^{**}(t). \tag{191}$$

Substituting $\mathrm{E}_t^Q[\delta_1(T)]$ and $\mathrm{E}_t^Q[\delta_3(T)]$ into equation (189)

$$\begin{aligned}
&\mathrm{E}_t^Q[\delta_2(T)] \\
&= \frac{\mathrm{E}_t^Q[\delta(T)] - p_1\mathrm{E}_t^Q[\delta_1(T)] - (1-p_1-p_2)\mathrm{E}_t^Q[\delta_3(T)]}{p_2} \\
&= \frac{\frac{\sum_{n=0}^{\infty}\frac{(\lambda(T-t))^n e^{-\lambda(T-t)}}{n!}\Phi(h_{1,n}(t))}{d(t)\sum_{n=0}^{\infty}\frac{(\lambda T)^n e^{-\lambda T}}{n!}\Phi(-h_{2,n}(t))} - p_1 + \frac{\sum_{n=0}^{\infty}\frac{(\lambda(T-t))^n e^{-\lambda(T-t)}}{n!}\left[p_1\Phi\big(-h_{2,n}^*(t)\big) - \frac{\Phi(h_{1,n}^*(t))}{d(t)}\right]}{\sum_{n=0}^{\infty}\frac{(\lambda(T-t))^n e^{-\lambda(T-t)}}{n!}\Phi(-h_{2,n}(t))}}{p_2} \\
&+ \frac{-(1-p_1-p_2) + \frac{\sum_{n=0}^{\infty}\frac{(\lambda(T-t))^n e^{-\lambda(T-t)}}{n!}\left[\frac{\Phi(h_{1,n}^{**}(t)) - \Phi(h_{1,n}(t))}{d(t)} + \Phi(-h_{2,n}(t)) - (p_1+p_2)\Phi\big(-h_{2,n}^{**}(t)\big)\right]}{\sum_{n=0}^{\infty}\frac{(\lambda(T-t))^n e^{-\lambda(T-t)}}{n!}\Phi(-h_{2,n}(t))}}{p_2} \\
&= 1 - \frac{\sum_{n=0}^{\infty}\frac{(\lambda(T-t))^n e^{-\lambda(T-t)}}{n!}\left[\frac{\Phi(h_{1,n}^*) - \Phi(h_{1,n}^{**}(t))}{d(t)} + (p_1+p_2)\Phi\left(-h_{2,n}^{**}(t)\right) - p_1\Phi\left(-h_{2,n}^*(t)\right)\right]}{p_2\sum_{n=0}^{\infty}\frac{(\lambda(T-t))^n e^{-\lambda(T-t)}}{n!}\Phi(-h_{2,n}(t))}
\end{aligned} \tag{192}$$

Substituting $\mathrm{E}_t^Q[\delta_1(T)]$ and $\mathrm{E}_t^Q[\delta_2(T)]$ into equation (168)

$$\frac{\frac{\sum_{n=0}^{\infty}\frac{(\lambda(T-t))^n e^{-\lambda(T-t)}}{n!}\left[\Phi\big(-h_{2,n}^*(t)\big) - \frac{\Phi(h_{1,n}^*(t))}{d(t)p_1}\right]}{\sum_{n=0}^{\infty}\frac{(\lambda(T-t))^n e^{-\lambda(T-t)}}{n!}\Phi(-h_{2,n}(t))}}{\frac{\sum_{n=0}^{\infty}\frac{(\lambda T)^n e^{-\lambda T}}{n!}\left[\frac{\Phi(h_{1,n}^*) - \Phi(h_{1,n}^{**}(t))}{d(t)} + (p_1+p_2)\Phi\big(-h_{2,n}^{**}(t)\big) - p_1\Phi\big(-h_{2,n}^*(t)\big)\right]}{p_2\sum_{n=0}^{\infty}\frac{(\lambda T)^n e^{-\lambda T}}{n!}\Phi(-h_{2,n}(t))}} = \frac{P(t,T) - V_1(t,T)}{P(t,T) - V_2(t,T)}. \tag{193}$$

After rearrangement, we have

$$\begin{aligned}
&\frac{\sum_{n=0}^{\infty}\frac{(\lambda(T-t))^n e^{-\lambda(T-t)}}{n!}\left[p_1\Phi\left(-h_{2,n}^*(t)\right) - \frac{\Phi(h_{1,n}^*(t))}{d(t)}\right]}{\sum_{n=0}^{\infty}\frac{(\lambda T)^n e^{-\lambda T}}{n!}\left[\frac{\Phi(h_{1,n}^*) - \Phi(h_{1,n}^{**}(t))}{d(t)} + (p_1+p_2)\Phi\left(-h_{2,n}^{**}(t)\right) - p_1\Phi\left(-h_{2,n}^*(t)\right)\right]} \\
&\quad = \frac{p_1[P(t,T) - V_1(t,T)]}{p_2[P(t,T) - V_2(t,T)]}.
\end{aligned} \tag{194}$$

Substituting $E_t^Q[\delta_1(T)]$ and $E_t^Q[\delta_3(T)]$ into equation (169)

$$\frac{\frac{\sum_{n=0}^{\infty}\frac{(\lambda(T-t))^n e^{-\lambda(T-t)}}{n!}\left[\Phi\left(-h_{2,n}^{*}(t)\right)-\frac{\Phi(h_{1,n}^{*}(t))}{d(t)p_1}\right]}{\sum_{n=0}^{\infty}\frac{(\lambda(T-t))^n e^{-\lambda(T-t)}}{n!}\Phi(-h_{2,n}(t))}}{\frac{\sum_{n=0}^{\infty}\frac{(\lambda T)^n e^{-\lambda T}}{n!}\left[\frac{\Phi(h_{1,n}^{**}(t))-\Phi(h_{1,n}(t))}{d(t)}+\Phi(-h_{2,n}(t))-(p_1+p_2)\Phi\left(-h_{2,n}^{**}(t)\right)\right]}{(1-p_1-p_2)\sum_{n=0}^{\infty}\frac{(\lambda T)^n e^{-\lambda T}}{n!}\Phi(-h_{2,n}(t))}} = \frac{P(t,T)-V_1(t,T)}{P(t,T)-V_3(t,T)}. \tag{195}$$

After rearrangement, we have

$$\frac{\sum_{n=0}^{\infty}\frac{(\lambda(T-t))^n e^{-\lambda(T-t)}}{n!}\left[p_1\Phi\left(-h_{2,n}^{*}(t)\right)-\frac{\Phi(h_{1,n}^{*}(t))}{d(t)}\right]}{\sum_{n=0}^{\infty}\frac{(\lambda T)^n e^{-\lambda T}}{n!}\left[\frac{\Phi(h_{1,n}^{**}(t))-\Phi(h_{1,n}(t))}{d(t)}+\Phi(-h_{2,n}(t))-(p_1+p_2)\Phi\left(-h_{2,n}^{**}(t)\right)\right]}$$
$$= \frac{p_1[P(t,T)-V_1(t,T)]}{(1-p_1-p_2)[P(t,T)-V_3(t,T)]}. \tag{196}$$

There are three unknown parameters λ, μ_Y, and σ_Y in equation (194) and equation (196). With one more equation (160)

$$\frac{V(t,T)}{P(t,T)} = \sum_{n=0}^{\infty}\frac{(\lambda(T-t))^n e^{-\lambda(T-t)}}{n!}\left\{\frac{\Phi(h_{1,n}(t))}{d(t)}+\Phi\left(h_{2,n}(t)\right)\right\} \tag{197}$$

where $V(t,T) = p_1V_1(t,T)+p_2V_2(t,T)+(1-p_1-p_2)V_3(t,T)$, we can solve the parameters numerically. The Q-measure probability of default and recovery rate can thus be uniquely derived.

In the case where a firm issued bonds consists of five seniority classes, the jump frequency and mean and variance of the normal jump should be consistent using any three of the seniority classes or arbitrage opportunities exist. When traders notice arbitrage opportunities, they will trade in the market so that the opportunities will disappear or the no arbitrage assumption will be violated.

An alternative method to calculated the parameters is through grouping. If we treat the class 2, 3, and 4 as one class, then there are totally three regrouped classes. Class 1 and 5 can stay the same. However, the new class 2 will represent $(p_2+p_3+p_4)$ proportion of the total debt. The mean recovery rate for the new class 2 is thus given by

$$E[\delta_2^{new}(T)] = \frac{p_2E[\delta_2(T)]+p_3E[\delta_3(T)]+p_4E[\delta_4(T)]}{p_2+p_3+p_4} \tag{198}$$

and a mean bond price is given by

$$V_2^{new}(t,T) = \frac{p_2V_2(t,T)+p_3V_3(t,T)+p_4V_4(t,T)}{p_2+p_3+p_4}. \tag{199}$$

5 Conclusion and future research

Both structural models and reduced-form model are both based on the no-arbitrage assumption where the bond price is regarded as an contingent claim under risk neutral valuation. If the first passage time is defined as $\tau = \inf\{t \mid V_t < B\}$, then an structural model defines an equivalent an intensity process. In this paper, we work on the models where the default can happen at the maturity of the bond. By pointing out mean recovery rate $\frac{\Phi(h_1)}{d\Phi(-h_2)}$ and default probability $\Phi(-x_2)$, we show that the Merton (1974) credit spread formula $R(t,T) - r = -\frac{1}{\tau}\ln\left\{\Phi[h_2(t)] + \frac{1}{d}\Phi[h_1(t)]\right\}$ is equivalent to the reduced-form yield spread formula (61) $R(t,T) - r(t,T) = -\frac{1}{T-t}\ln\left\{1 - \{1 - \mathrm{E}_t^Q[\delta(\tau)]\}\Pr\{\tau \leq T\}\right\}$. The separation of default probability or intensity from mean recovery rate might not be achieved without further assumption either on the value or the firm, intensity, or recovery rate.

Zhou (1998) extends the Merton (1974) model by adding a jump process to the diffusion process to include the possibility of nonsystematic surprise. The degree of freedom are increased at least by three, one from the drift of the diffusion process, one from the jump frequency, and at least one from jump size. When the jump represents nonsystematic risk, the systematic information does not appear in the jump process that the instantaneous returns are forced to equal riskless rate and the jump process with its compensator has a zero mean return. This eliminates one degree of the freedom. If a firm issue bonds of different seniorities, the investors reflex their risk preference towards the jump frequency and jump sizes through the prices of bonds of different seniorities. The relationship of the mean recovery rate between different seniority classes is given be (167) $\left\{1 - \mathrm{E}_t^Q[\delta_i(T)]\right\}\Pr\{V_T < B\} = \frac{P(t,T) - V_i(t,T)}{P(t,T)} \quad i = 1, 2, 3$. We can solve the parameters through numerical method.

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