Abstract: Let \((X_1, Y_1), (X_2, Y_2), \ldots,\) be bivariate claim sizes arising from some insurance portfolio. The number of claims occurring during time interval \([0, t]\) is denoted by \(N(t)\) with \(N(\cdot)\) some point process on \([0, \infty)\). We investigate in this short paper distributional and asymptotic properties of the following point process

\[
C_t(B) := \sum_{i=1}^{N(t)} 1\left( (X_{N(t)}, Y_{N(t)}) - (X_i, Y_i) \in B \right), \quad t \geq 0,
\]

with \((X_{N(t)}, Y_{N(t)})\) the bivariate maximum insurance claim occurring during \([0, t]\) and \(B\) some Borel set of \([0, \infty)^2\).

We show among others that \(C_t(\cdot)/t, C_t(\cdot)/N(t)\) can be used for statistical purposes to estimate upper tail probability of the claim size distribution.

Further, based on extreme value theory we investigate connection between convergence in distribution of the bivariate maximum claim size and weak convergence of our point process.

Of special interest is the implication of the weak convergence result for ECOMOR reinsurance.

Key Words: Near bivariate maximum insurance claim; Multivariate extreme value theory; Order statistics, Weak convergence; Almost sure convergence; Central limit theorem; ECOMOR reinsurance.


*I would like to thank Allianz Suisse for kindly financing my participation in 6th Congress in Insurance: Mathematics and Economics.
1 Introduction

Let \((X_1, Y_1), (X_2, Y_2), \ldots\) be independent bivariate random vectors with continuous distribution function \(F\) on \(\mathbb{R}^2\). In a specific insurance context, \(X_i\) may for instance stand for total claim amount from accident \(i\), whereas \(Y_i\) the total expenses needed to settle the claim. Assume further that \(N(\cdot)\) is a point process on \([0, \infty)\) so that \(\{N(t), t \geq 0\}\) is independent of claim sizes \((X_i, Y_i), i \geq 1\) and starting at time 0 almost surely at 1.

Pakes and Li (2001) consider application of near maxima results (obtained in previous papers) to insurance for a univariate scenario. Realising the importance of multivariate modelling, we extend in this short note results of Hashorva and Hüsler (2001) employing the same approach as Pakes and Li (2001) to the bivariate setup.

Using standard notations we write in the sequel, \((X_{n:n}, Y_{n:n})\) for the componentwise maxima, and \((X_{N(t):N(t)}, Y_{N(t):N(t)})\) if we consider a random sample size instead.

The main object of study here is the following point process driven by the bivariate maximum insurance claim

\[
\mathcal{C}_t(B) := \sum_{i=1}^{N(t)} \mathbf{1}((X_{N(t):N(t)} - X_i, Y_{N(t):N(t)} - Y_i) \in B),
\]

with \(B\) a Borel set in \(\mathcal{B}([0, \infty)^2)\).

In other words, we are interested in the number of claim sizes occurring in the time interval \([0, t]\) that fall in random regions defined by sample maxima.

There are several aspects that make this point process interesting: first we may get an impression for the magnitude of maxima itself (as \(n\) increases), since intuitively for fixed \(B \in \mathcal{B}([0, \infty)^2)\) the ratio \(C_n(B)/n\) will vanish if maxima increases to infinity.

Secondly, strongly consistent estimators for certain tail probabilities can be constructed. More precisely, we show that \(\mathcal{C}_t(B)/t, \mathcal{C}_t(B)/N(t)\) are consistent estimator of special tail probabilities related to \(B\). Further, by establishing Central Limit Theorem for these estimators, we may easily construct confidence intervals for the estimated quantity.

The third result is obtained with respect to ECOMOR reinsurance treaty. To obtain that we consider the relation between convergence in distribution of sample maxima and weak convergence of the point process of interest, which is an interesting result per se.

Organisation of the rest of the paper: In the next section we derive distributional results. Then in a separate section we establish almost sure convergence and CLT results. The third sections deals with weak convergence of the point process. An application of those results is given in a separate section. Few comments and remarks conclude this article.

2 Distributional Results

We assume throughout that \(N(t)\) is almost surely finite for all \(t > 0\) with finite expectation. It is well-known that the probability generating function (p.g.f.) is a compact tool that summarises all information concerning the distribution function of the underlying random variable. It is for instance an easy task to obtain all moments, by differentiating the p.g.f. and taking limit.
In the next proposition we obtain the p.g.f. of \( C_t(B) \).

**Proposition 2.1.** Let \( B \in \mathcal{B}([0, \infty)^2) \) and \( s \in (0, 1) \), \( t > 0 \) then the probability generating function of \( C_t(B) \) is given by

\[
E \{ s^{C_t(B)} \} = E \left\{ N(t)[s1(0 \in B) + 1(0 \notin B)]E \left\{ p(s, X_1, Y_1, B)^{N(t)-1} \right\} \right\} \\
+ E \left\{ N(t)(N(t) - 1)E \left\{ 1(X_1 > X_2, Y_2 > Y_1)1((X_1 - X_2, 0) \notin B, (0, Y_2 - Y_1) \notin B) \right\} \\
+ s1((X_1 - X_2, 0) \in B, (0, Y_2 - Y_1) \notin B) + 1((X_1 - X_2, 0) \notin B, (0, Y_2 - Y_1) \in B) \right\} \\
+ s^21((X_1 - X_2, 0) \in B, (0, Y_2 - Y_1) \in B) \right\} \right\},
\]

with

\[
p(s, x, y, B) := F(x, y) - (1 - s)P\{(x - X_3, y - Y_3) \in B\}, \quad (x, y) \in \mathcal{R}^2.
\]

**Proof of Proposition 2.1.:** Let in the following

\[
p(x, y, B) := \int_{(x-u,y-v) \in B} dF(u, v), \quad p^*(x, y, B) := F(x, y) - p(x, y, B).
\]

By conditioning we may write for \( n \geq 1, 0 \leq j \leq n \)

\[
P\{C_t(B) = j|N(t) = n\}
= nP\left\{ \sum_{i=1}^{n} 1((X_{n:n} - X_i, Y_{n:n} - Y_i) \in B) = j, X_{n:n} = X_1, Y_{n:n} = Y_1 \right\}
+ n(n-1)P\left\{ \sum_{i=1}^{n} 1((X_{n:n} - X_i, Y_{n:n} - Y_i) \in B) = j, X_{n:n} = X_1, Y_{n:n} = Y_2 \right\}
= nP\left\{ \sum_{i=2}^{n} 1((X_{n:n} - X_i, Y_{n:n} - Y_i) \in B) = j - 1(0 \in B), \ X_i \leq X_1, Y_i \leq Y_1, i = 2, \ldots, n \right\}
+ n(n-1)P\left\{ \sum_{i=3}^{n} 1((X_{n:n} - X_i, Y_{n:n} - Y_i) \in B) = j, (X_1 - X_2, 0) \notin B, (0, Y_2 - Y_1) \notin B \right\}
\]
\[
X_i \leq X_1, Y_i \leq Y_2, i = 1, \ldots, n
+ n(n-1)P\left\{ \sum_{i=3}^{n} 1((X_{n:n} - X_i, Y_{n:n} - Y_i) \in B) = j - 1, (X_1 - X_2, 0) \notin B, (0, Y_2 - Y_1) \in B \right\}
\]
\[
X_i \leq X_1, Y_i \leq Y_2, i = 1, \ldots, n
+ n(n-1)P\left\{ \sum_{i=3}^{n} 1((X_{n:n} - X_i, Y_{n:n} - Y_i) \in B) = j - 2, (X_1 - X_2, 0) \in B, (0, Y_2 - Y_1) \notin B \right\}
\]
\[
X_i \leq X_1, Y_i \leq Y_2, i = 1, \ldots, n
+ n(n-1)P\left\{ \sum_{i=3}^{n} 1((X_{n:n} - X_i, Y_{n:n} - Y_i) \in B) = j - 2, (X_1 - X_2, 0) \in B, (0, Y_2 - Y_1) \in B \right\}
\]
\[
X_i \leq X_1, Y_i \leq Y_2, i = 1, \ldots, n
= \frac{n!}{(n-j)!(j-1(0 \in B))!}E\{p(X_1, Y_1, B)^{j-1(0 \in B)}p^*(X_1, Y_1, B)^{n-j}\}
\]

3
\begin{align*}
&+ \frac{n!}{(n-j-2)!j!} E\{1(X_1 > X_2, Y_2 > Y_1)1((X_1 - X_2, 0) \notin B, (0, Y_2 - Y_1) \notin B) \\
&\times p(X_1, Y_2, B)^j p^*(X_1, Y_2, B)^{n-j-2}\}
\end{align*}
\begin{align*}
&+ \frac{n!}{(n-j-1)!} E\left\{1(X_1 > X_2, Y_2 > Y_1)\left[1((X_1 - X_2, 0) \notin B, (0, Y_2 - Y_1) \notin B) \right. \\
&\left. +1((X_1 - X_2, 0) \in B, (0, Y_2 - Y_1) \notin B)\right] p(X_1, Y_2, B)^{j-1} p^*(X_1, Y_2, B)^{n-j-1}\right\}
\end{align*}
\begin{align*}
&+ \frac{n!}{(n-j)!} E\{1(X_1 > X_2, Y_2 > Y_1)1((X_1 - X_2, 0) \in B, (0, Y_2 - Y_1) \in B) \\
&\times p(X_1, Y_2, B)^{j-2} p^*(X_1, Y_2, B)^{n-j}\},
\end{align*}

where the above terms are 0 if they are not defined and 0! := 1.

Since for \(s \in (0,1), t > 0\) we have
\[
E\{s^C_t(B)\} = \sum_{k=1}^{\infty} \sum_{i=0}^{k} s^i P\{C_t(B) = i|N(t) = k\} P\{N(t) = k\}
\]
the proof follows easily employing the binomial formula. \(\square\)

In view of the above formula for the probability generating function, we may easily derive moments of \(C_t(B)\). A compact expression for expectation of \(E\{C_t(B)\}\) is obtained by straightforward calculations writing
\[
E\{C_t(B)\} = E\{E\{C_t(B)|N(t)\}\}
= E\{N(t) P\{(X_{N(t):N(t)}, Y_{N(t):N(t)}) - (X_1, Y_1) \in B\}\}
= E\{N(t) p(X_{N(t):N(t)}, Y_{N(t):N(t)}, B)\}. \tag{2.2}
\]
Clearly expectation is finite if \(E\{N(t)\}\) is finite. We assume throughout in the sequel that \(E\{N(t)\} < \infty, t > 0\). Now, if \(t\) goes to infinity, it is natural to require that
\[
N(t) \overset{p}{\to} \infty, \quad t \to \infty. \tag{2.3}
\]
Note that by the monotonicity, the above convergence holds almost surely. A stronger condition is to require
\[
\frac{N(t)}{t} \overset{a.s.}{\to} Z, \quad t \to \infty, \tag{2.4}
\]
with \(Z\) a positive random variable.

Though standard notations, we mention that \(\overset{p}{\to}, \overset{a.s.}{\to}, \overset{d}{\to}, \overset{w}{\to}, \overset{v}{\to}\) stand for convergence in probability, almost surely, in distribution, weak and vague convergence (of measures), respectively. Further we would like to mention that both assumptions on \(N(t)\) were previously considered in Pakes and Li (2001).

Let \(\omega_1, \omega_2\) denote the upper endpoints of the support of the marginal distribution function \(F_1, F_2\) respectively, and put \(\omega := (\omega_1, \omega_2)\). It is well-known that almost surely
\[
(X_{n:n}, Y_{n:n}) \to \omega, \quad n \to \infty. \tag{2.5}
\]
In general considering \(X_{N(t):N(t)}, Y_{N(t):N(t)}\) instead of \(X_{n:n}, Y_{n:n}\) requires further knowledge of the behaviour of the integer valued random variable \(N(t)\). The following lemma turns out to be very helpful in our context:
Lemma 2.2. (Embrechts et al. (1997)) Let $U_1, U_2, \ldots$ a sequence of random variables so that

$$U_n \xrightarrow{a.s.} U, \quad n \to \infty.$$ 

If $N(t), t \geq 0$ is independent of $U_i, i \geq 1$ such that $N(t) \xrightarrow{a.s.} \infty (N(t) \xrightarrow{p} \infty)$ then

$$U_{N(t)} \xrightarrow{a.s.} (\xrightarrow{p}) Z, \quad t \to \infty. \quad (2.6)$$

When $U_n \xrightarrow{d} U$ and $N(t) \xrightarrow{p} \infty$ then

$$U_{N(t)} \xrightarrow{d} U \quad (2.7)$$

holds as $t \to \infty$.

Proof of Lemma 2.2.: See Lemma 2.5.3 and Lemma 2.5.6 of Embrechts et al. (1997) for a proof.

A simple proof of (2.7) can be obtained as follows: for given $\varepsilon > 0$ we get by the convergence in distribution for $k, n > k'$

$$|\mathbb{P}\{U_{N(t)} \leq i\} - \mathbb{P}\{U_n \leq i\}| \leq \sum_{k=1}^{\infty} |\mathbb{P}\{U_k \leq i\} - \mathbb{P}\{U_n \leq i\}| \mathbb{P}\{N(t) = k\} \leq \varepsilon \mathbb{P}\{N(t) > k'\} + 2k' \mathbb{P}\{N(t) \leq k'\}, \quad i \in \mathbb{N},$$

thus letting $t \to \infty$ the claim follows. \hfill $\Box$

Consequently, if we assume (2.3) then

$$(X_{N(t):N(t)}, Y_{N(t):N(t)}) \xrightarrow{a.s.} \omega, \quad t \to \infty$$

holds. If further (2.4) is satisfied and the random sequence $N(t)/t$ is uniformly integrable (it implies that $\sup_{t \geq 0} \mathbb{E}\{N(t)\} < \infty$ and further $\mathbb{E}\{Z\} < \infty$) then by Lebesgue dominated convergence theorem

$$\lim_{t \to \infty} \frac{\mathbb{E}\{C_t(B)\}}{t} = \mathbb{P}\{\omega - (X_1, Y_1) \in B\} \mathbb{E}\{Z\} < \infty$$

holds. Note in passing that if at least one component of $\omega$ is equal to $\infty$ then the rhs above is 0.

We show in the next section how to estimate the tail probability $p(B) := \mathbb{P}\{(X_1, Y_1) \in \omega - B\}$. Clearly if $\max(\omega_1, \omega_2) = \infty$ the mentioned probability is equal 0, hence if our estimation result is significantly different from 0, then we not only estimate the probability of interest, but also may conclude that the support of underlying distribution function $F$ is bounded from above.

Furthermore, if $F$ is known in a special parametrised form (see example below), then estimators for the parameters can be obtained using the estimated value of $p(B)$ on special Borel sets $B$.

3 Estimators of $p(B)$

Almost sure convergence of $C_n(B)/n$ was shown in Hashorva and Hüsler (2001) for some special Borel sets. It means that we have a strongly consistent estimator of the probability of interest $p(B)$.

In this section we generalise that result for Borel sets $B \subset [0, \infty)^2$, and derive additionally almost sure convergence of $C_t(B)/N(t), C_t(B)/t$, the estimators for the insurance model. The latter is achieved by imposing some mild restrictions on the counting random variable $N(t)$. Further we consider a CLT result for the proposed estimator.
Proposition 3.1. Let \((X_i, Y_i), i \geq 1\) iid bivariate random vectors with common continuous distribution function \(F\). Then we have for \(B\) a Borel set in \([0, \infty)^2\)

\[
\lim_{n \to \infty} C_n(B) = P\{\omega - (X_1, Y_1) \in B\}, \tag{3.8}
\]

If further \(N(t)\) satisfies the limit relation in (2.4) being independent of \((X_i, Y_i), i \in \mathcal{N}\), then

\[
\lim_{t \to \infty} \frac{C_t(B)}{N(t)} = P\{\omega - (X_1, Y_1) \in B\}, \tag{3.9}
\]

\[
\lim_{t \to \infty} \frac{C_t(B)}{t} = ZP\{\omega - (X_1, Y_1) \in B\}, \tag{3.10}
\]

hold almost surely.

Proof of Proposition 3.1.: By Lemma 2.2 it suffices to show only the first claim. The third claim is immediate, since

\[
C_t(B) = C_t(B) N(t) \frac{N(t)}{t}, \quad t > 0
\]

and (2.4) yields the claim.

Note in passing that in view of (2.5) we have \(p(B) = 0\) if \(\omega_i = \infty\) for \(i = 1\) or \(i = 2\).

Now, we show initially the claim for rectangular Borel sets \(B = [0, a] \times [0, b]\), with \(a, b\) positive constants. Assume without loss of generality that \(p(B) < 1\). For \(\varepsilon > 0\) sufficiently small we have

\[
P\left\{\frac{C_n(B)}{n} > p(B) + \varepsilon, \text{i.o.}\right\} = P\left\{\bigcap_{i=1}^{\infty} \bigcup_{n \geq i} \{C_n(B) > n(p(B) + \varepsilon)\}\right\}
\]

\[
= P\left\{\bigcap_{i=1}^{\infty} \bigcup_{n \geq i} \{X_{n:n} - X_{n-[n(p(B)+\varepsilon)]}:n \leq a, Y_{n:n} - Y_{n-[n(p(B)+\varepsilon)]}:n \leq b\}\right\}
\]

\[
= P\left\{X_{n:n} - X_{n-[n(p(B)+\varepsilon)]}:n \leq a, Y_{n:n} - Y_{n-[n(p(B)+\varepsilon)]}:n \leq b, \text{i.o.}\right\}
\]

\[
= 0
\]

by strong consistency of bivariate empirical quantile function. Similarly we get

\[
P\left\{\frac{C_n(B)}{n} > p(B) - \varepsilon, \text{i.o.}\right\} = 1,
\]

hence

\[
\limsup_{n \to \infty} \frac{C_n(B)}{n} = p(B)
\]

almost surely and along the same lines we get

\[
\liminf_{n \to \infty} \frac{C_n(B)}{n} = p(B),
\]

almost surely. So we conclude that \(C_n(B)/n\) converges almost surely to \(p(B)\).

For some general Borel set \(B\) we need to show the claim only for the case that upper endpoints of the support of marginal distributions \(F_1, F_2\) are finite.
Fix again some \( \varepsilon \) positive. Then we have for \( n \geq 2 \)

\[
\frac{C_n(B)}{n} \leq \sum_{i=1}^{n} 1((X_{n,i} - X_i, Y_{n,i} - Y_i) \in B, X_{n,i} > \omega_1 - \varepsilon, Y_{n,i} > \omega_2 - \varepsilon) + 1(X_{n,i} \leq \omega_1 - \varepsilon) + 1(Y_{n,i} \leq \omega_2 - \varepsilon)
\]

\[
\leq \sum_{i=1}^{n} 1((X_i, Y_i) \in \omega - B - [0, \varepsilon]^2) + 1(X_{n,i} \leq \omega_1 - \varepsilon) + 1(Y_{n,i} \leq \omega_2 - \varepsilon). \tag{3.11}
\]

Next, in light of (2.5), the second term above converges to 0 almost surely. By Strong Law of Large Numbers letting further \( \varepsilon \to 0 \) we get

\[
\limsup_{n \to \infty} \frac{C_n(B)}{n} \leq p(B)
\]

almost surely.

One way to complete the proof is to construct a lower bound for \( C_n(B)/n \). If \( B \) is a rectangular box this follows easily.

Alternatively we show that the class of Borel sets \( \mathcal{L} \) such that \( \liminf_{n \to \infty} C_n(B)/n \geq p(B) \) holds almost surely for all \( B \in \mathcal{L} \) is a \( \lambda \)-system. Since almost sure convergence holds for all Borel sets \( B \) that are rectangular boxes as shown above, this property holds by the monotone class theorem for all Borel sets \( B \), hence the proof. \( \square \)

**Proposition 3.2.** Let \((X_i, Y_i), i \geq 1 \) iid bivariate random vectors with common continuous distribution function \( F \) such that \( \omega \) has finite components. Let \( B_i, i \leq k \) be rectangular boxes in \([0, \infty)^2\) and put \( \xi_{n,i} := \sqrt{n}(C_n(B_i)/n - \mathbb{P}\{\omega - (X_1, Y_1) \in B_i\}) \). Assume that there exists a bivariate sequence with positive elements \((\varepsilon_{11}, \varepsilon_{12}), (\varepsilon_{21}, \varepsilon_{22}), \ldots\), tending to \((0, 0)\) such that

\[
\lim_{n \to \infty} n \log F_i(\omega_i - \varepsilon_{n,i}) = -\infty, \quad i = 1, 2 \tag{3.12}
\]

and further for all \( i = 1, \ldots, k \)

\[
\lim_{n \to \infty} \sqrt{n} \mathbb{P}\left\{(X_1, Y_1) \in \left(\omega - [0, \varepsilon_{n1}] \times [0, \varepsilon_{n2}] - B_i\right) \setminus \left(\omega - B_i\right)\right\} = 0. \tag{3.13}
\]

Then we have

\[
\left(\xi_{n,1}, \ldots, \xi_{n,k}\right) \xrightarrow{d} (W_1, \ldots, W_k), \quad n \to \infty, \tag{3.14}
\]

with \( W_i \) centered Gaussian random variables so that

\[
\text{Cov}\{W_i, W_j\} = \mathbb{P}\{\omega - (X_1, Y_1) \in (B_i \cap B_j)\} - \mathbb{P}\{\omega - (X_1, Y_1) \in B_i\} \mathbb{P}\{\omega - (X_1, Y_1) \in B_j\}.
\]

If further \( N(t) \) satisfies the limit relation in (2.3) and is independent of the claim sizes, then also

\[
\left(\xi_{N(t),1}, \ldots, \xi_{N(t),k}\right) \xrightarrow{d} (W_1, \ldots, W_k), \quad n \to \infty. \tag{3.15}
\]

**Proof of Proposition 3.2.:** By the assumptions and the continuous mapping theorem it suffices to show the proof for \( B_i := [0, a_i] \times [0, b_i], a_i, b_i \in (0, \infty), i = 1, \ldots, k \). We make use of Cramer-Wood device; let therefore \((t_1, \ldots, t_k) \in (0, \infty)^k\) and consider

\[
\eta_n := \sum_{i=1}^{k} t_i C_n(B_i)/n, \quad n \geq 2.
\]

7
Similarly to (3.11) we have
\[ nn_{\eta n} \leq \sum_{i=1}^{k} \sum_{i=1}^{n} t_i 1((X_i, Y_i) \in \omega - B_i - [0, \varepsilon_{n1}] \times [0, \varepsilon_{n2}]) \]
\[ + n \left[ 1(X_{n:n} \leq \omega_1 - \varepsilon_{n1}) + 1(Y_{n:n} \leq \omega_2 - \varepsilon_{n2}) \right] \sum_{i=1}^{k} t_i. \]

Hence recalling the assumptions
\[ 0 \leq \frac{1}{\sqrt{n}} \left[ nn_{\eta n} - \sum_{j=1}^{k} \sum_{i=1}^{n} t_j 1((X_i, Y_i) \in \omega - B_j) \right] \]
\[ \leq \frac{1}{\sqrt{n}} \sum_{j=1}^{k} \sum_{i=1}^{n} t_j 1 \left( (X_i, Y_i) \in \left( \omega - B_j - [0, \varepsilon_{n1}] \times [0, \varepsilon_{n2}] \right) \setminus \omega - B_j \right) \]
\[ + \sqrt{n} \left[ 1(X_{n:n} \leq \omega_1 - \varepsilon_{n1}) + 1(Y_{n:n} \leq \omega_2 - \varepsilon_{n2}) \right] \sum_{j=1}^{k} t_j \]
\[ = o_p(1), \quad n \to \infty. \]

Thus the first claim follows easily by the CLT for iid random variables. The second claim follows by applying again Lemma 2.2. \qed

**Example.** Let \((X_i, Y_i), i \geq 1\) iid bivariate random vectors with uniformly distributed marginal distributions and joint distribution function
\[ F(x, y) = \rho xy + (1 - \rho) \min(x, y), \quad (x, y) \in [0, 1]^2. \]

Further assume that \(N(t)\) is a homogenous Poisson process with positive parameter \(\lambda\). Clearly \(N(t) \to \infty\) almost surely.

Let us consider \(C_t(B)\) with Borel set \(B = ([0, a] \times [0, 1]) \cup ([0, 1] \times [0, a])\). Simple calculations and the above results yield
\[ \frac{C_t(B)}{N(t)} \overset{a.s.}{\to} 1 - (1 - a)(1 - \rho a), \quad t \to \infty \]
and since \(N(t)/t \overset{a.s.}{\to} \lambda\) we have
\[ \frac{C_t(B)}{t} \overset{a.s.}{\to} \lambda[1 - (1 - a)(1 - \rho a)], \quad t \to \infty. \]

Further the conditions of Proposition 3.2 hold. Take for example \(\varepsilon_{n1} = \varepsilon_{n2} = n^{-c}, c > 1, n \geq 2\).

It is easy to check the result of the above example by simulations. We performed several simulations in Splus 6, varying \(\rho, a\) and \(\lambda\), which confirmed our findings.

### 4 Connection with weak convergence of maximum claim

Our point process of interest \(C_t(B)\) is governed to a large extend by the behaviour of maximum claim \((X_{N(t);N(t)}, Y_{N(t);N(t)})\). The latter is intrinsically related with \((X_{n:n}, Y_{n:n}), n \in \mathbb{N}\). Multivariate extreme value theory deals with asymptotic behaviour of linearly transformed maxima. In the sequel
we rely to a large extend on multivariate extreme value theory for iid samples. Assume therefore that there exists \( q_1(t), q_2(t) \) two positive scaling functions and \( r_1(t), r_2(t) \) so that

\[
\left( \frac{X_{n:n} - r_1(n)}{q_1(n)}, \frac{Y_{n:n} - r_2(n)}{q_2(n)} \right) \xrightarrow{d} (\mathcal{X}_1, \mathcal{Y}_1)
\]

as \( n \to \infty \). This implies that there exists a max-stable bivariate distribution function \( H \) such that

\[
\lim_{t \to \infty} \sup_{(x,y) \in \mathbb{R}^2} \left| F^H(t(x,y)) - H(x,y) \right| = 0,
\]

with

\[
T_t(x,y) := (q_1(t)x + r_1(t), q_2(t)y + r_2(t)).
\]

It is important to note that the marginal distributions \( H_i, i = 1, 2 \) can be one of the following extreme value distributions

\[
H_i(x) = \begin{cases} 
\exp(- \exp(-x)), & x \in \mathcal{R}, \\
\exp(-(-x)^\alpha), & \alpha > 0, \ x \in (-\infty, 0], \\
\exp(-x^{-\alpha}), & \alpha > 0, \ x \in (0, \infty)
\end{cases}
\]

the so called standard Gumbel, Weibull or Fréchet distribution, see Embrechts et al. (1997) or Falk et al. (1994) for more details. We drop the word standard in the sequel.

Convergence in distribution for \( C_t(B) \) can be treated by investigating the limit behaviour of the probability generating function previously derived for \( t \to \infty \).

Bringing in the scene the weak convergence of maxima, we may consider instead weak convergence of \( C_t(B_t) \), with \( B_t := (q_1(t), q_2(t))B \), thereby allowing \( B_t \) to vary with \( t \) such that the scaling for the weak convergence of maxima is being retrieved.

In the special case that \( q_1(t), q_2(t) \) are constant, say equal 1, which implies that \( H_1, H_2 \) are both Gumbel, we retrieve weak convergence for \( C_t(B) \).

A clearer picture of weak convergence results is shown by the next proposition:

**Proposition 4.1.** Assume that (4.17) holds with \( H \) a bivariate max-stable distribution function so that \( H(\mathbf{x}) > 0 \) for all \( \mathbf{x} \in \mathbb{R}^2 \). If further (2.4) holds then for any Borel set \( B \subset [0, \infty)^2 \)

\[
\mathcal{C}_t(B_t) \xrightarrow{d} C(B)
\]

where the random variable \( C(B) \) has probability generating function given by

\[
\mathbb{E}\{s^{C(B)}\} = \mathbb{E}\left\{ \int_{(x,y) \in \mathbb{R}^2} \exp(-(1-s)Z\mu\{(x,y) - B\})H^Z(x,y)\,d\mu(x,y) \right\}
\]

\[
+ Z^2 \int_{(x,y,u,v) \in \mathbb{R}^4} \left[ \mathbf{1}\{(0,v-y) \notin B, (x-u,0) \notin B\} + s \mathbf{1}\{(0,v-y) \in B, (x-u,0) \notin B\} \right.
\]

\[
+ s \mathbf{1}\{(0,v-y) \notin B, (x-u,0) \in B\} + s^2 \mathbf{1}\{(0,v-y) \in B, (x-u,0) \in B\} \right]
\]

\[
\times \exp(-(1-s)Z\mu\{(x,v) - B\})H^Z(x,v)\,d\mu(x,v)\,d\mu(u,v),
\]

with \( \mu \) the Radon measure generated by the set function \( \ln H(\mathbf{x}, \mathbf{y}), (\mathbf{x}, \mathbf{y}) \subset \mathbb{R}^2 \) and \( s \in (0,1) \).

Furthermore, if both marginal distributions \( H_1, H_2 \) are standard Gumbel then (4.19) holds with 1 instead of \( Z \).
Proof of Proposition 4.1.: For \( t > 0 \) let \( \mu_t \) denote the measure generated by the set function
\[
\ln F(T_t(x, y), T_t(u, v)), \quad (x, y) \times (u, v) \subset \mathcal{R}^2,
\]
(see Corollary 2.26 of Kallenberg for set functions and corresponding measures on \( \mathcal{R}^d, d \geq 1 \)). Further let \( E := [l, \infty) \setminus \{l\} \) with \( l = \inf[H > 0] \). Clearly \( E \) is not empty. By (4.17) the vague limit on \( E \) of \( \mu_t \) as \( t \to \infty \) exists and is a unique Radon measure defined by the set function \( \ln H((x, y), (u, v)) \), with \( (x, y) \times (u, v) \subset \mathcal{R}^2 \), i.e.
\[
\mu_t \xrightarrow{v} \mu, \quad t \to \infty.
\]
We refer the reader for more details about the limit measure \( \mu \) (also known as exponential measure) to Resnick (1987). Now (4.17) implies for all \( (x, y) \in \mathcal{R}^2 \)
\[
\lim_{t \to \infty} F^d(T_t(x, y)) = H(x, y),
\]
locally uniformly, hence for \( z > 0 \)
\[
\lim_{t \to \infty} \left[ F(T_t(x, y)) - (1 - s)P\{T_t(x, y) - (X_1, Y_1) \in B_t\} \right]^{lz}
\]
\[
= \left( \lim_{t \to \infty} F^{lz}(T_t(x, y)) \exp(-(1 - s)z \lim_{t \to \infty} tP\{(X_1, Y_1) \in (q_1(t), q_2(t))[(x, y) - B] + (r_1(t), r_2(t))\}) \right)
\]
\[
= 1((x, y) : \mu\{(x, y) - B \} < \infty) H^z(x, y) \exp(-(1 - s)z\mu\{(x, y) - B\})
\]
holds locally uniformly on \( \mathcal{R}^2 \). Since by the assumption the distribution function \( H \) has Weibull or Gumbel marginal distributions
\[
1((x, y) : \mu\{(x, y) - B\} < \infty) = 1, \quad (x, y) \in \mathcal{R}^2.
\]
Using now the expression for the probability generating function and applying further Theorem 3.27 of Kallenberg (1997) we may write
\[
E \{s^{L_t(B_t)}\}
\]
\[
= E \left\{ N(t) \int_{\mathcal{R}^2} [s1(0 \in B_t) + 1(0 \notin B_t)] \left[ F(x, y) - (1 - s)P\{(x - X_2, y - Y_2) \in B_t\} \right]^{N(t)-1} dF(x, y) \right\}
\]
\[
+ E \left\{ (N(t)(N(t) - 1))/2 \int_{\mathcal{R}^4} [1((0, v - y) \notin B_t, (x - u, 0) \notin B_t) + s1((0, v - y) \notin B_t, (x - u, 0) \in B_t) + s^21((0, v - y) \in B_t, (x - u, 0) \in B_t)] \times \left[ F(x, v) - (1 - s)P\{(x - X_3, v - Y_3) \in B_t\} \right]^{N(t)-2} dF(x, y) dF(u, v) \right\}
\]
\[
= [s1(0 \in B) + 1(0 \notin B)] E \left\{ N(t)/t \int_{\mathcal{R}^2} \left[ F(T_t(x, y)) - (1 - s)P\{T_t(x, y) - (X_2, Y_2) \in B_t\} \right]^{t(N(t)-1)/t} \times d\mu_t(x, y) \right\} + E \left\{ (N(t)(N(t) - 1))/2 \int_{\mathcal{R}^4} [1((0, v - y) \notin B, (x - u, 0) \notin B) + s1((0, v - y) \in B, (x - u, 0) \notin B) + s^21((0, v - y) \in B, (x - u, 0) \in B)] \left[ F(T_t(x, v)) - (1 - s)P\{T_t(x, v) - (X_3, Y_3) \in B_t\} \right]^{t(N(t)-2)/t} \times d\mu_t(x, y) d\mu_t(u, v) \right\}
\]
\[ \rightarrow [s1(0 \in B) + 1(0 \notin B)]E \left\{ Z \int_{(x,y) \in \mathbb{R}^2} H^Z(x, y) \exp(-(1-s)Z\mu \{(x, y) - B\}) d\mu(x, y) \right\} \]

\[ + E \left\{ Z^2 \int_{(x,y,u,v) \in \mathbb{R}^4} 1((0, v - y) \notin B, (x - u, 0) \notin B) \]

\[ + s1((0, v - y) \in B, (x - u, 0) \notin B) + s1((0, v - y) \notin B, (x - u, 0) \in B) \]

\[ + s^21((0, v - y) \in B, (x - u, 0) \in B) \exp(-(1-s)Z\mu \{(x, v) - B\})H^Z(x, v) d\mu(x, y) d\mu(u, v) \right\}. \]

By Problem 2.8 p.80 of Reiss (1989) we have for all Borel sets \( B \subset \mathbb{R}^2 \)

\[ P\{(X_{n:n}, Y_{n:n}) \in B\} = \int_B nF^{n-1}(x) dF(x) + \int_B n(n-1)F^{n-2}(x)F_1(x_1|x_2)F_2(x_2|x_1) d(F(x_1, \infty) \times F(\infty, x_2)), \]

with

\[ F_1(x|y) := P\{X_1 \leq x | Y_1 = y\}, \quad F_2(y|x) := P\{Y_1 \leq y | X_1 = x\}. \]

So employing further this fact it follows along the lines of Hashorva and Hüsler (2001) that

\[ \lim_{s \rightarrow 1-} E\{s^{C(B)}\} = E\left\{ \int_{\mathbb{R}^2} dH^Z(x, y) \right\}. \]

Clearly, the rhs above is 1 since both lower endpoints of the support of marginal distributions \( H_1, H_2 \)

are equal to \(-\infty\).

Next we show that if both marginal distributions \( H_1, H_2 \) are standard Gumbel then the random variable \( Z \) in the limit probability generating function can be substituted by 1. Recall first the well-known homogeneity property of some bivariate max-stable distributions function \( G \) with standard Fréchet marginal distributions, which means that

\[ G^t(tx, ty) = G(x, y) \]

holds for all \( t, x, y > 0 \). In our case we obtain for \( z > 0 \)

\[ H^z(x, y) = H^z(\ln u, \ln v) \]

\[ =: H^z_u(u, v) \]

\[ = H^z(u/z, v/z) \]

\[ = H_*(u/z, v/z) \]

\[ = H(\ln(u/z), \ln(v/z)) \]

\[ = H(x - \ln z, y - \ln z), \]

since the bivariate max-stable distribution function \( H_* \) has standard Fréchet marginal distributions.

Hence by the definition also

\[ z\mu(x, y) = \mu(x - \ln z, y - \ln z], \]

thus the proof follows easily by the properties of the integral. \( \square \)

**Remark 4.2.** a) If one of the marginal distributions \( F_1 \) or \( F_2 \) is in the max-domain of attraction of Fréchet distribution then \( C_1((q_1(t), q_2(t))B) \) is not tight for \( B \) Borel sets containing a rectangular box. Hence no weak convergence for the point process \( C_1(\cdot) \) can be stated.

b) It is possible to show convergence in distributions under weaker assumptions, for example by dropping the independence assumption concerning the claim sizes; details for univariate setup can be found in Hashorva (2001).
The above result shows that the marginal distributions of the point process
\[ C^*_t(B) := C_t((q_1(t), q_2(t))B) \]
converge weakly for \( t \to \infty \). It can be further shown along the same lines of the proof above that
the joint weak convergence of the marginals of the point process \( C^*_t(\cdot) \) hold, establishing so weak
convergence to a point process \( C(B) \). In the next proposition we highlight the case \( F \) has both
marginal distribution in the max-domain of attraction of a Gumbel random variable.

**Proposition 4.3.** Under the assumptions of the previous proposition, if
\( H_i(t) = \exp(-\exp(-t)), i = 1, 2 \) then
\[ C^*_t(\cdot) \xrightarrow{w} C(\cdot) \quad (4.20) \]
in \((0, \infty)^2\) with \( C(\cdot) \) a mixed Cox process with stochastic intensity
\[ \nu((a, b), (U_1, U_2)) = \log \left( \frac{H((U_1, U_2) - a)H((U_1, U_2) - b)}{H((U_1, U_2) - (a_1, b_2))H((U_1, U_2) - (b_1, a_2))} \right), \quad 0 < a < b, a, b \in \mathbb{R}^2 \]
where \((U_1, U_2)\) is a bivariate random vector with underlying continuous distribution function \( H \).

**Proof of Proposition 4.3.** Let \( B_1, B_2, \ldots, B_k \) be Borel sets in \((0, \infty)^2\). Then along the same lines
of the previous proposition we get
\[ (C^*_t(B_1), \ldots, C^*_t(B_k)) \xrightarrow{w} (C(B_1), \ldots, C(B_k)), \]
as \( t \to \infty \). We claim that the limit distributions do not depend on \( Z \). This can be established along
the same lines of proof of Theorem 4.1 in Pakes and Li (2001).

Hence we may assume that \( N(t) \) is almost surely equal \( t + 1 \), implying thus \( Z = 1 \). So the weak limit of
\( C^*_t(\cdot) \) is the same as the weak limit of the point process \( \sum_{i=1}^n 1((X_{n:n} - X_i, Y_{n:n} - Y_i) \in (q_1(n), q_2(n))B) \)
obtained in Theorem 2.5 of Hashorva and Hüsler (2001), consequently the claim follows by the result
of the aforementioned proposition. \( \square \)

5 ECOMOR Reinsurance

In this section we investigate implications of the weak convergence results for the ECOMOR reinsurance
treaty. Let \( (X_i, Y_i), i \geq 1 \) be iid claims with underlying distribution function \( F \) independent of
the counting process \( N(t) \). Assume for illustration purposes that our claims arise from some motor
insurance portfolio where \( X_i \) denotes the amount of claim from a particular accident, whereas \( Y_i \)
denotes the amount of expenses associated with the \( i \)th claim.

Let us suppose that reinsurance cover is available for the claim amount in excess of the \( p \)th largest
claim from the time that the contract is written \( t = 0 \) up to some fixed time say \( t \), which is given by
\[ L_p(t) := \sum_{i=1}^{N(t)} (X_{N(t)-i+1:N(t)} - X_{N(t)-p+1:N(t)})^+, \quad p \geq 2, t > 0 \]
if \( N(t) > p \) and \( L_p(t) := 0 \) otherwise.
Further suppose that a similar treaty

$$L_q(t) := \sum_{i=1}^{N(t)} (Y_{N(t)-i+1:N(t)} - Y_{N(t)-q+1:N(t)})^+, \quad q \geq 2, t > 0$$

if $N(t) > q$ and $L_q(t) := 0$ otherwise, is available covering the expenses with respect to the $q$th largest claim.

Realising the central role of the upper order statistics, we relate their weak convergence with weak convergence for $(L_p(t), L_q(t))$ as $t \to \infty$.

Suppose therefore that (4.17) holds, then it follows (see Embrechts et al. (1997)) that

$$\left( (X_{n:n} - r(n))/q(n), \ldots, (X_{n-k+1:n} - r(n))/q(n) \right) \overset{d}{\to} (X_1, \ldots, X_k)$$

$$\left( (Y_{n:n} - r(n))/q(n), \ldots, (Y_{n-k+1:n} - r(n))/q(n) \right) \overset{d}{\to} (Y_1, \ldots, Y_k)$$

where the random vector $(X_1, \ldots, X_k)$ and $(Y_1, \ldots, Y_k)$ have density function

$$I_k(x) = H(x) \prod_{j=1}^k \frac{H'(x_j)}{H_i(x_j)}, \text{ with } x_1 < x_2 < \cdots < x_k, \quad x : \prod_{j=1}^k H_i(x_j) > 0,$$

with $i = 1, 2$ respectively. Moreover the weak convergence above holds jointly.

Now using if (2.3) holds, then joint weak convergence is retrieved

$$\left( \left( \frac{X_{N(t):N(t)} - r_1(t)}{q_1(t)}, \frac{Y_{N(t):N(t)} - r_2(t)}{q_2(t)} \right), \ldots, \left( \frac{X_{N(t)-k+1:N(t)} - r_1(t)}{q_1(t)}, \frac{Y_{N(t)-k+1:N(t)} - r_2(t)}{q_2(t)} \right) \right)$$

$$\overset{d}{\to} \left( (X_1^*, Y_1^*), \ldots, (X_k^*, Y_k^*) \right).$$

The interesting case is when we assume that $F$ has both components in the max-domain of attraction of Gumbel distribution. In view of Proposition 4.3 we get

$$\left( X_i^* - X_j^*, Y_i^* - Y_j^* \right) \overset{d}{\to} \left( X_i - X_j, Y_i - Y_j \right), \quad 1 \leq i < j, \quad i, j \in \mathbb{N},$$

hence we get

$$\left( \frac{L_p(t)}{q_1(t), L_q(t)} / q_2(t) \right)$$

$$= \left( \sum_{i=1}^{p-1} \frac{X_{N(t)-i+1:N(t)} - X_{N(t)-p+1:N(t)}}{q_1(t)}, \sum_{i=1}^{q-1} \frac{Y_{N(t)-i+1:N(t)} - Y_{N(t)-q+1:N(t)}}{q_2(t)} \right)$$

$$\overset{d}{\to} \left( \sum_{i=1}^{p-1} \left( \frac{X_{N(t)-i+1:N(t)} - r_1(t)}{q_1(t)}, \frac{X_{N(t)-p+1:N(t)} - r_1(t)}{q_1(t)} \right), \right.$$

$$\left. \sum_{i=1}^{q-1} \left( \frac{Y_{N(t)-i+1:N(t)} - r_2(t)}{q_2(t)}, \frac{Y_{N(t)-q+1:N(t)} - r_2(t)}{q_2(t)} \right) \right)$$

$$\overset{d}{\to} \left( \sum_{i=1}^{p-1} (X_i^* - X_p^*), \sum_{i=1}^{q-1} (Y_i^* - Y_q^*) \right) \text{ as } t \to \infty.$$
\[
\begin{align*}
&= \left( \sum_{i=1}^{p-1} i(\mathcal{X}_i - \mathcal{X}_{i+1}), \sum_{i=1}^{q-1} i(\mathcal{Y}_i - \mathcal{Y}_{i+1}) \right) \\
&\overset{d}{=} \left( \sum_{i=1}^{p-1} i(\mathcal{X}_i - \mathcal{X}_{i+1}), \sum_{i=1}^{q-1} i(\mathcal{Y}_i - \mathcal{Y}_{i+1}) \right). 
\end{align*}
\]

It follows further (shown in Hashorva and Hüsler (2000c)) that as \( t \to \infty \)
\[
\sum_{i=1}^{p-1} i(\mathcal{X}_i - \mathcal{X}_{i+1}) \overset{d}{=} \sum_{i=1}^{p-1} E_{i,p}
\]
and
\[
\sum_{i=1}^{q-1} i(\mathcal{Y}_i - \mathcal{Y}_{i+1}) \overset{d}{=} \sum_{i=1}^{q-1} E_{i,q},
\]
with \( E_{1,p}, E_{2,p}, \ldots \) and \( E_{1,q}, E_{2,q} \ldots \) iid standard exponential random variables.

Under asymptotic independence, meaning that \( H \) is a product distribution we have further that \( E_{i}^{p} \) are independent of \( E_{i}^{q}, i \in \mathcal{N} \) so we may write for this case
\[
\left( L_p(t)/q_1(t), L_q(t)/q_2(t) \right) \overset{d}{\to} \left( \sum_{i=1}^{p-1} E_{i,p}, \sum_{i=1}^{q-1} E_{i,q} \right)
\]
as \( t \to \infty \). This does not hold in general if asymptotic dependence occurs.

The conclusion of this section is that if we assume Gumbel marginal distributions for \( H \) then it implies that the positive random variable \( Z \) does not influence the asymptotic distribution of the scaled total amount of each treaty.

### 6 Further comments and remarks

Typically in insurance models used by actuaries \( N(t) \) is some renewal (counting) process defined as follows
\[
N(t) := \sup \left\{ j : \sum_{i=1}^{j} U_i \leq t \right\}, \quad t \geq 0,
\]
with \( U_i, i \geq 1, \) iid positive random variables. It is well known that
\[
\frac{N(t)}{t} \xrightarrow{a.s.} \frac{1}{E\{U_1\}}, \quad t \to \infty,
\]
which holds also if \( E\{U_1\} = \infty \), interpreting \( 1/\infty = 0 \).

Indeed, condition (2.4) is satisfied for all renewal processes such that \( E\{U_1\} \) is finite.

Remark that the assumption \( V_i, i \in \mathcal{N} \) are iid exponentially distributed implies \( N(t) \) is a homogenous Poisson process; it is by far the most common stochastic process encountered in insurance applications dealing with count data.

Also for mixed Poisson process condition (2.4) is satisfied.
Now, the strong consistency of $C_t(B)/t$ and CLT result can be used for various estimation purposes. Clearly, there are already many estimators in the literature, however we did not make any comparisons. Based on simulation study for special bivariate distribution functions the result was satisfactory.

Another important result is the weak convergence of the marginals of the point process. It is implied by the assumption on the weak convergence of maxima, which is quite common and reasonable to require for practical applications. It is worth noting that instead of assuming joint weak convergence in (4.16) we may consider directly conditions that imply joint weak convergence of (5.21).

It is interesting to consider also point processes that are driven by other upper order statistics; the univariate case is treated in Hashorva and Hüsler (2000b), Hashorva and Hüsler (2000c). This topic will be discussed in a forthcoming paper.

Acknowledgement: I would like to thank Professor Jürg Hüsler for motivating discussions and helpful remarks and Mr. Roberto Bianchi, Chief of Actuarial Department, Allianz Suisse for kind support and encouragement.

References


