Geometric Brownian Motion Models for Assets and Liabilities: From Pension Funding to Optimal Dividends

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1. Introduction

Consider a business with asset value $A(t)$ and liability value $L(t)$ at time $t$, $t \geq 0$. For example, these could be the assets and liabilities of a pension fund, or the assets and liabilities of a firm. In an ideal world, assets would always be sufficient to meet liabilities. But this is not possible due to their stochastic nature. As a consequence, appropriate measures have to be taken.

Here, we propose the following solution. It is postulated that the value of the assets should always be in a band (or corridor) that is defined by $L_1(t)$ and $L_2(t)$, with $L_1(0) \leq A(0) \leq L_2(0)$ and $L_1(t) < L_2(t)$, $t \geq 0$. For example, $L_1(t)$ could be 99 percent of $L(t)$ and $L_2(t)$ 105 percent of $L(t)$. Whenever the asset value threatens to fall below $L_1(t)$, sufficient funds are provided to prevent this from happening. If, on the other hand, the asset value is about to exceed $L_2(t)$, the assets are reduced by the potential overflow. In the context of a pension plan, the sponsor of the plan provides a guarantee at the level $L_1(t)$, and he is entitled to withdraw funds at the level $L_2(t)$. Note that the original asset value $A(t)$ is now replaced by the modified asset value $A_m(t)$. In the language of stochastic processes, the process $\{A_m(t)\}$ is obtained from the process $\{A(t)\}$ by introduction of the retaining (stochastic) barriers at $\{L_1(t)\}$ and $\{L_2(t)\}$.

A goal of this paper is to calculate $V_1(\alpha; \lambda_1, \lambda_2)$, $0 < \lambda_1 \leq \alpha \leq \lambda_2$, the expectation of the present value of the payments that have to be provided at the guaranteed level $L_1(t)$, and $V_2(\alpha; \lambda_1, \lambda_2)$, $0 < \lambda_1 \leq \alpha \leq \lambda_2$, the expectation of the present value of the “overflow” at the level $L_2(t)$. Here $\alpha = A(0)$ and $\lambda_j = L_j(0)$, $j = 1, 2$, denote the initial values.
In particular we are interested in situations where $V_1 = V_2$, that is, where the overflow at the upper barrier can fund the guarantee at the lower barrier. A detailed analysis is given in Section 7, which concludes the first part of the paper.

The second part of the paper starts with Section 8. In the geometric Brownian motion model for assets and liabilities, we assume two barriers as in the first part. At the upper barrier, there is an overflow, as in the first part, which is now interpreted as the dividends according to a barrier strategy. However, if the (modified) asset values fall to the liability value (which is the lower barrier), “ruin” takes place, and no more dividends can be paid. We derive an explicit expression for $V(\alpha; \lambda_1, \lambda_2)$, the expected discounted dividends (before ruin). From this we find an explicit expression for $\lambda_2^*$, the value of $\lambda_2$ that maximizes $V(\alpha; \lambda_1, \lambda_2)$. A particularly simple and intriguing expression is found for $V_2(\lambda_2^*; \lambda_1, \lambda_2^*)$. This expression can be interpreted as the present value of a deterministic perpetuity with exponentially growing payments.

2. The Model

We assume that asset and liability values are geometric Brownian motions of the form:

$$A(t) = \alpha e^{X(t)},$$

$$L(t) = \lambda e^{Y(t)},$$

with \{X(t), Y(t)\} being a bivariate Brownian motion (Wiener process), with $X(0) = Y(0) = 0$, drift parameters $\mu_X$ and $\mu_Y$, instantaneous variances $\sigma_X^2$ and $\sigma_Y^2$, and correlation $\rho$. We shall examine the case where the barriers are multiples of $L(t)$, that is,

$$L_j(t) = \lambda_j e^{Y(t)}, \quad j = 1, 2,$$
with $0 < \lambda_1 < \lambda_2$. It is assumed that between the barriers the modified asset value process $\{A^m(t)\}$ has the same instantaneous rate of return as the original asset value process $\{A(t)\}$.

Present values are calculated with respect to a valuation force of interest $\delta$. The expectation of the present value of $L_j(t)$ is

$$e^{-\delta t}E[L_j(t)] = \lambda_j e^{-(\delta - \delta_j) t}, \quad (2.4)$$

where

$$\delta_j = \ln E[e^{Y(1)}] = \mu_Y + \frac{1}{2} \sigma_Y^2, \quad (2.5)$$

is the liability growth rate. The payments are made whenever the modified asset values are on one of the two barriers. Hence, to assure the existence of $V_1$ and $V_2$, we assume that the integral of (2.4) is finite, i.e., that

$$\delta_Y < \delta. \quad (2.6)$$

We shall also use the notation

$$\delta_X = \ln E[e^{X(1)}] = \mu_X + \frac{1}{2} \sigma_X^2, \quad (2.7)$$

which is the asset growth rate, and

$$\sigma^2 = \text{Var}[X(1) - Y(1)] = \sigma_X^2 - 2\rho \sigma_X \sigma_Y + \sigma_Y^2. \quad (2.8)$$

In this model, the functions $V_1(\alpha; \lambda_1, \lambda_2)$ and $V_2(\alpha; \lambda_1, \lambda_2)$ are homogeneous of degree 1, that is, for each $z > 0$,

$$V_j(z\alpha; z\lambda_1, z\lambda_2) = zV_j(\alpha; \lambda_1, \lambda_2), \quad j = 1, 2. \quad (2.9)$$
This can be seen by a change of the monetary unit.

3. The First Visit at a Barrier

Let

\[ T = \min\{t \geq 0 \mid A(t) = L_1(t) \text{ or } A(t) = L_2(t)\} \]  \hspace{1cm} (3.1)

be the first time when the asset value reaches one of the two barriers. Note that no payments are made before time \( T \). Let \( I(.) \) denote the indicator function of an event. Then

\[ V_j(\alpha; \lambda_1, \lambda_2) = E[e^{-\delta T} V_j(A(T); L_1(T), L_2(T))] \]

\[ = E[e^{-\delta T} I(A(T) = L_1(T)) V_j(A(T); L_1(T), L_2(T))] \]

\[ + E[e^{-\delta T} I(A(T) = L_2(T)) V_j(A(T); L_1(T), L_2(T))]. \]  \hspace{1cm} (3.2)

Using the homogeneity property of the function \( V_j \) and the fact that \( L_1(T)/L_2(T) = \lambda_1/\lambda_2 \), we see that

\[ V_j(\alpha; \lambda_1, \lambda_2) = E[e^{-\delta T} I(A(T) = L_1(T)) A(T)] V_j(1; 1, \lambda_2/\lambda_1) \]

\[ + E[e^{-\delta T} I(A(T) = L_2(T)) A(T)] V_j(1; \lambda_1/\lambda_2, 1). \]  \hspace{1cm} (3.3)

Motivated by this formula, we introduce the following pair of functions

\[ J_1(\alpha; \lambda_1, \lambda_2) = E[e^{-\delta T} I(A(T) = L_1(T)) A(T)], \]  \hspace{1cm} (3.4)

\[ J_2(\alpha; \lambda_1, \lambda_2) = E[e^{-\delta T} I(A(T) = L_2(T)) A(T)]. \]  \hspace{1cm} (3.5)

Note that each can be interpreted as an expected present value of a contingent payment of \( A(T) \) at time \( T \). Now formula (3.3) can be written as

\[ V_j(\alpha; \lambda_1, \lambda_2) = J_1(\alpha; \lambda_1, \lambda_2) V_j(1; 1, \lambda_2/\lambda_1) + J_2(\alpha; \lambda_1, \lambda_2) V_j(1; \lambda_1/\lambda_2, 1). \]  \hspace{1cm} (3.6)

In the next section we shall find explicit expressions for \( J_1 \) and \( J_2 \).
4. The Functions \( J_1 \) and \( J_2 \)

To determine the functions \( J_1 \) and \( J_2 \), we consider martingales of the form

\[
\{e^{-\delta t + \theta X(t) + (1-\theta)Y(t)}, \; t \geq 0\}.
\]

(4.1)

The process (4.1) is a martingale, provided that

\[
1 = e^{-\delta} E[e^{\theta X(1) + (1-\theta)Y(1)}],
\]

or

\[
0 = -\delta + E[\theta X(1) + (1-\theta)Y(1)] + \frac{1}{2} \text{Var}[\theta X(1) + (1-\theta)Y(1)]
\]

\[
= -\delta + \mu_X \theta + \mu_Y (1-\theta) + \frac{\sigma_X^2}{2} \theta^2 + \frac{\sigma_Y^2}{2} (1-\theta)^2 + \rho \sigma_X \sigma_Y \theta (1-\theta),
\]

(4.2)

which is a quadratic equation for \( \theta \). Note that, for \( \theta = 0 \), the right-hand side of (4.2) is

\[
-\delta + \mu_Y + \frac{\sigma_Y^2}{2} = -\delta + \delta_Y,
\]

which is negative by condition (2.6). It follows that equation (4.2) has a positive solution \( \theta_+ \) and a negative solution \( \theta_- \). Furthermore, for \( \theta = 1 \), the right-hand side of (4.2) equals

\[
-\delta + \delta_X.
\]

Hence

\[
\theta_+ > 1 \; \text{if} \; \delta > \delta_X,
\]

\[
\theta_+ = 1 \; \text{if} \; \delta = \delta_X,
\]

\[
0 < \theta_- < 1 \; \text{if} \; \delta < \delta_X.
\]

(4.3)

For \( \theta = \theta_+ \) or \( \theta_- \), the process (4.1) is a martingale. If we stop it at time \( T \) and use the optional sampling theorem, we obtain

\[
1 = E[e^{-\delta T + \theta X(T) + (1-\theta)Y(T)}]
\]

\[
= E[e^{[X(T) - Y(T)](\theta - 1)} e^{-\delta T + X(T)} I(A(T) = L_1(T))]
\]

\[
+ E[e^{[X(T) - Y(T)](\theta - 1)} e^{-\delta T + X(T)} I(A(T) = L_2(T))]
\]
\[
\frac{1}{\alpha} \left( \frac{\lambda_1}{\alpha} \right)^{\theta-1} J_1(\alpha; \lambda_1, \lambda_2) + \frac{1}{\alpha} \left( \frac{\lambda_2}{\alpha} \right)^{\theta-1} J_2(\alpha; \lambda_1, \lambda_2),
\]

or

\[
\alpha^\theta = \lambda_1^{\theta-1} J_1(\alpha; \lambda_1, \lambda_2) + \lambda_2^{\theta-1} J_2(\alpha; \lambda_1, \lambda_2). \tag{4.4}
\]

With \( \theta = \theta_+ \) and \( \theta = \theta_- \), we thus obtain two linear equations for \( J_1 \) and \( J_2 \). It is judicious to write their solutions in the following form. For \( 0 < \lambda_1 \leq \alpha \leq \lambda_2 \),

\[
J_1(\alpha; \lambda_1, \lambda_2) = \lambda_1 \frac{g(\alpha / \lambda_2)}{g(\lambda_1 / \lambda_2)}, \tag{4.5}
\]

\[
J_2(\alpha; \lambda_1, \lambda_2) = \lambda_2 \frac{g(\alpha / \lambda_1)}{g(\lambda_2 / \lambda_1)}, \tag{4.6}
\]

with the auxiliary function

\[
g(z) = z^{\theta_+} - z^{\theta_-}. \tag{4.7}
\]

5. A Differential Equation Approach

In this section we present an alternative derivation of (4.5) and (4.6). It provides additional insight.

In the following, \( K(\alpha; \lambda_1, \lambda_2) \) can be \( J_1(\alpha; \lambda_1, \lambda_2), J_2(\alpha; \lambda_1, \lambda_2), V_1(\alpha; \lambda_1, \lambda_2) \) or \( V_2(\alpha; \lambda_1, \lambda_2) \). For \( 0 \leq t < T \), the conditional expectation of the random variable

\[
e^{-\delta T} K(A(T); L_1(T), L_2(T)),
\]

given the information up to time \( t \), is a martingale. As there are no payments between time 0 and time \( T \), the martingale is the process

\[
\{e^{-\delta t} K(A(t); L_1(t), L_2(t)); 0 \leq t < T\}. \tag{5.1}
\]

We shall show that this implies that the function \( K \) satisfies the second-order order differential equation
\[ \frac{\sigma^2}{2} \alpha^2 K''(\alpha; \lambda_1, \lambda_2) + (\delta_X - \delta_Y)\alpha K'(\alpha; \lambda_1, \lambda_2) + (-\delta + \delta_Y)K(\alpha; \lambda_1, \lambda_2) = 0. \quad (5.2) \]

The Appendix gives an alternative derivation of (5.2) using Itô calculus.

Because of the homogeneity property of \( K \),

\[ e^{-\delta t} K(A(t); L_1(t), L_2(t)) = e^{-\delta t + Y(t)} K(\alpha e^{X(t)} - Y(t); \lambda_1, \lambda_2). \quad (5.3) \]

Hence we have the condition

\[ E[e^{-\delta dt + Y(dt)} K(\alpha e^{X(dt)} - Y(dt); \lambda_1, \lambda_2)] = K(\alpha; \lambda_1, \lambda_2). \quad (5.4) \]

Neglecting terms whose expectation is of order smaller than \( dt \), we can write

\[ e^{-\delta dt + Y(dt)} = 1 - \delta dt + Y(dt) + \frac{1}{2}[Y(dt)]^2 \quad (5.5) \]

and

\[ K(\alpha e^{X(dt)} - Y(dt); \lambda_1, \lambda_2) = K(\alpha; \lambda_1, \lambda_2) \]

\[ + K'(\alpha; \lambda_1, \lambda_2)\alpha\{X(dt) - Y(dt) + \frac{1}{2}[X(dt) - Y(dt)]^2\} \]

\[ + \frac{1}{2} K''(\alpha; \lambda_1, \lambda_2)\alpha^2[X(dt) - Y(dt)]^2. \quad (5.6) \]

The product of (5.5) and (5.6) is

\[ K(\alpha; \lambda_1, \lambda_2)\{1 - \delta dt + Y(dt) + \frac{1}{2}[Y(dt)]^2\} \]

\[ + \alpha K'(\alpha; \lambda_1, \lambda_2)\{X(dt) - Y(dt) + \frac{1}{2}[X(dt) - Y(dt)]^2 + [X(dt) - Y(dt)]Y(dt)\} \]

\[ + \frac{1}{2} \alpha^2 K''(\alpha; \lambda_1, \lambda_2)[X(dt) - Y(dt)]^2, \]

the expectation of which is

\[ K(\alpha; \lambda_1, \lambda_2)(1 - \delta dt + \delta_Y dt) + \alpha K'(\alpha; \lambda_1, \lambda_2)(\delta_X dt - \delta_Y dt) \]

\[ + \frac{1}{2} \alpha^2 K''(\alpha; \lambda_1, \lambda_2)\sigma^2 dt, \quad (5.7) \]
where $\delta_Y$, $\delta_X$ and $\sigma$ are defined by (2.5), (2.7) and (2.8), respectively. Equating (5.7) with the right-hand side of (5.4), subtracting $K(\alpha; \lambda_1, \lambda_2)$ on both sides, and canceling $dt$ yields the differential equation (5.2). The general solution of this homogeneous linear differential equation is of the form

$$K(\alpha; \lambda_1, \lambda_2) = C_1\alpha^{\theta_1} + C_2\alpha^{\theta_2}, \quad (5.8)$$

where $\theta_1$ and $\theta_2$ are solutions of the so-called indicial equation

$$\frac{\sigma^2}{2}\theta(\theta - 1) + (\delta_X - \delta_Y)\theta + (-\delta + \delta_Y) = 0. \quad (5.9)$$

This quadratic equation is equivalent to (4.2). Hence, $\theta_2 = \theta_+$ and $\theta_1 = \theta_-$, for example.

The coefficients $C_1$ and $C_2$ in (5.8) are determined from the boundary conditions. For example, for $K(\alpha; \lambda_1, \lambda_2) = J_1(\alpha; \lambda_1, \lambda_2)$, they are

$$J_1(\lambda_1; \lambda_1, \lambda_2) = \lambda_1, \quad (5.10)$$

$$J_1(\lambda_2; \lambda_1, \lambda_2) = 0. \quad (5.11)$$

Now it is easy to verify that expression (4.5) is a linear combination of $\alpha^{\theta_+}$ and $\alpha^{\theta_-}$ and satisfies the boundary conditions (5.10) and (5.11).

6. The Functions $V_1$ and $V_2$

Now we are prepared to calculate the expected present value of the cashflows at the two barriers, $V_1$ and $V_2$. It follows from (3.6) and (4.5) – (4.7), or more directly from (5.8) with $K = V_j$, that $V_j(\alpha; \lambda_1, \lambda_2)$ is a linear combination of $\alpha^{\theta_+}$ and $\alpha^{\theta_-}$. The coefficients are determined from the following boundary conditions:

$$V_1'(\lambda_1; \lambda_1, \lambda_2) = -1, \quad (6.1)$$

$$V_1'(\lambda_2; \lambda_1, \lambda_2) = 0, \quad (6.2)$$
The solutions are best expressed in terms of the auxiliary function

\[ h(z) = \frac{z^\theta}{\theta_+} - \frac{z^\theta}{\theta_-}. \]  

(6.5)

Then it is easily verified that

\[ V_1(\alpha; \lambda_1, \lambda_2) = -\lambda_2 \frac{h(\alpha/\lambda_2)}{h'(\lambda_2/\lambda_2)} \]  

(6.6)

and

\[ V_2(\alpha; \lambda_1, \lambda_2) = \lambda_1 \frac{h(\alpha/\lambda_1)}{h'(\lambda_2/\lambda_1)} \]  

(6.7)

are linear combinations of \( \alpha^\theta_+ \) and \( \alpha^\theta_- \) and satisfy the conditions (6.1) – (6.4).

The heuristic justification of the boundary conditions (6.1) – (6.4) is as follows. If \( \alpha \) is “close” to \( \lambda_2 \), an increment of \( \Delta \alpha \) has roughly speaking no influence on the guaranteed payments at the lower barrier, which explains (6.2), but leads to an additional overflow of \( \Delta \alpha \) at the upper barrier, which explains (6.4). If, on the other hand, \( \alpha \) is “close” to \( \lambda_1 \), an increment of \( \Delta \alpha \) reduces the guaranteed payments at the lower barrier by \( \Delta \alpha \), which explains (6.1), but has no influence on the overflow at the upper barrier, which explains (6.3).

The two cases of a single barrier can be treated as limiting cases. If there is no upper barrier, the expectation of the present value of the payments that are made at the guaranteed level \( L_1(t) \) is obtained from (6.6) as the limit \( \lambda_2 \to \infty \). It is

\[ V_1(\alpha; \lambda_1, \infty) = \frac{\lambda_1}{-\theta_-} \left( \frac{\lambda_1}{\alpha} \right)^{-\theta_-}, \quad \lambda_1 \leq \alpha. \]  

(6.8)
Similarly, if there is no lower barrier, the expectation of the present value of the “overflow” at the barrier \( \lambda_2(t) \) is obtained from (6.7) as the limit \( \lambda_1 \to 0 \). It is

\[
V_2(\alpha; 0, \lambda_2) = \frac{\lambda_2}{\theta_1} \left( \frac{\alpha}{\lambda_2} \right) \theta_1^+, \quad \alpha \leq \lambda_2.
\] (6.9)

Formula (6.8) generalizes (2.13) [or (2.15) and (2.16)] in Gerber and Pafumi (2000). It is related to (2.17) in Gerber and Shiu (2002).

### 7. Cost-Neutral Asset Modification

A natural question is whether it is possible that the expected discounted value of the guaranteed payments (at the lower barrier) is equal to the expected discounted overflow at the upper barrier, that is, if

\[
V_1(\alpha; \lambda_1, \lambda_2) = V_2(\alpha; \lambda_1, \lambda_2)
\] (7.1)

for certain initial values \( \lambda_1 \leq \alpha \leq \lambda_2 \). To discuss this question, it is useful to have explicit versions of formulas (6.6) and (6.7) at our disposal:

\[
V_1(\alpha; \lambda_1, \lambda_2) = \frac{\lambda_1}{-\theta_1} \cdot \frac{\theta_1 \alpha^0 \lambda_1^0 - \theta_2 \alpha^0 \lambda_2^0}{\lambda_2^0 \lambda_1^0 - \lambda_1^0 \lambda_2^0},
\] (7.2)

\[
V_2(\alpha; \lambda_1, \lambda_2) = \frac{\lambda_2}{-\theta_1} \cdot \frac{\theta_1 \alpha^0 \lambda_1^0 - \theta_2 \alpha^0 \lambda_2^0}{\lambda_2^0 \lambda_1^0 - \lambda_1^0 \lambda_2^0}.
\] (7.3)

It follows that if (7.1) has a solution, then

\[
\alpha = \lambda_1 \left[ \frac{\theta_1}{\theta_2} \cdot \frac{\beta^{0,-1} - 1}{\beta^{0,-1} - 1} \right] ^{1 - \theta_2},
\] (7.4)

where

\[
\beta = \frac{\lambda_2}{\lambda_1}.
\] (7.5)
For further discussion, we distinguish two cases: (a) $\delta \leq \delta_X$ and (b) $\delta > \delta_X$.

(a) If $\delta \leq \delta_X$, we have $0 < \theta_+ \leq 1$ according to (4.3). Then (7.1) is not possible, because

$$V_1(\alpha; \lambda_1, \lambda_2) < V_2(\alpha; \lambda_1, \lambda_2)$$

(7.6)

for any $\alpha$. To show (7.6), we compare the numerators in (7.2) and (7.3). The coefficients of $\theta_+\alpha^\theta_-$ satisfy

$$\lambda_1 \lambda_2^{\theta_-} \leq \lambda_2 \lambda_1^{\theta_-},$$

(7.7)

because $\theta_+ \leq 1$. Similarly, the coefficients of $-\theta_+\alpha^\theta_-$ satisfy

$$\lambda_1 \lambda_2^{\theta_-} < \lambda_2 \lambda_1^{\theta_-},$$

(7.8)

because $\theta_- < 0$. Inequality (7.6) follows from (7.7) and (7.8).

(b) If $\delta > \delta_X$, we have $\theta_+ > 1$ according to (4.3). For given $0 < \lambda_1 < \lambda_2$, $V_1(\alpha; \lambda_1, \lambda_2)$ is a decreasing function of $\alpha$, and $V_2(\alpha; \lambda_1, \lambda_2)$ is an increasing function of $\alpha$. Hence, for given $0 < \lambda_1 < \lambda_2$, there is an $\alpha$, with $\lambda_1 \leq \alpha \leq \lambda_2$, for which (7.1) holds if and only if the following two conditions are satisfied:

$$V_1(\lambda_1; \lambda_1, \lambda_2) \geq V_2(\lambda_1; \lambda_1, \lambda_2),$$

(7.9)

$$V_1(\lambda_2; \lambda_1, \lambda_2) \leq V_2(\lambda_2; \lambda_1, \lambda_2),$$

(7.10)

or, equivalently, by the homogeneity of the functions $V_1$ and $V_2$,

$$V_1(1; 1, \beta) / V_2(1; 1, \beta) \geq 1,$$

(7.11)

$$V_2(\beta; 1, \beta) / V_1(\beta; 1, \beta) \geq 1,$$

(7.12)

with $\beta = \lambda_2 / \lambda_1$ as in (7.5). Substitution from (7.2) and (7.3) yields the equivalent conditions

$$\frac{\theta_+ \beta^{\theta_-} - \theta_- \beta^{\theta_-}}{\theta_+ - \theta_-} \geq 1,$$

(7.13)
\[
\frac{\theta_+ \beta^{\theta_+} - \theta_- \beta^{\theta_-}}{\theta_+ - \theta_-} \geq 1.
\]

(7.14)

Obviously, both (7.13) and (7.14) are satisfied for sufficiently large values of \(\beta\). In other words, (7.1) has a solution \(\alpha\) between \(\lambda_1\) and \(\lambda_2\) if \(\lambda_2\) is much larger than \(\lambda_1\). To investigate the problem in general, we consider the function

\[
\varphi(x) = \frac{\theta_+ x^{\theta_+ - 1} - \theta_- x^{\theta_- - 1}}{\theta_+ - \theta_-}, \quad x > 0.
\]

(7.15)

It has the property that

\[\varphi(1) = 1\]

and

\[\varphi'(1) = \theta_+ + \theta_- - 1.\]  
(7.16)

Rewriting the quadratic equation (5.9) as

\[
\frac{\sigma^2}{2} \theta^2 + (\delta_X - \delta_Y - \frac{\sigma^2}{2})\theta + (-\delta + \delta_Y) = 0,
\]

(7.17)

we see that

\[-\theta_+ - \theta_- = \frac{\delta_X - \delta_Y - \sigma^2/2}{\sigma^2/2},\]

or

\[\theta_+ + \theta_- - 1 = \frac{\delta_Y - \delta_X}{\sigma^2/2}. \]
(7.18)

Two subcases need to be distinguished: (b.1) \(\delta_X \neq \delta_Y\) and (b.2) \(\delta_X = \delta_Y\).

(b.1) If asset and liability growth rates are not equal, \(\delta_X \neq \delta_Y\), then it follows from (7.16) and (7.18) that \(\varphi'(1) \neq 0\). Thus for \(\beta\) sufficiently close to 1, either

\[\varphi(\beta) < 1\]

or
\( \phi(1/\beta) < 1. \)

In other words, one of the two conditions (7.13) and (7.14) is violated for values of \( \beta \) that are sufficiently close to 1.

(b.2) If asset and liability growth rates are equal, \( \delta_X = \delta_Y \), then \( \phi'(1) = 0 \). The function \( \phi(x) \) tends to \( \infty \) for \( x \) tending to 0 or \( \infty \). Because \( \phi'(x) = 0 \) only for \( x = 1 \), the function \( \phi(x) \) has its unique minimum at \( x = 1 \). This shows that (7.13) and (7.14) are satisfied for any \( \beta > 1 \). That is, in this subcase, (7.1) always has a solution \( \alpha \) between \( \lambda_1 \) and \( \lambda_2 \), as given by (7.4). Furthermore, applying \( \theta = 1 - \theta_+ \), we can rewrite the ratio within the brackets on the right-hand side of (7.4) as

\[
\frac{\beta^{\theta_+ - 1} - 1}{\theta_+ - 1} \cdot \frac{1 - \beta^{-\theta_+}}{\theta_+},
\]

whose Taylor series expansion at \( \beta = 1 \) begins with

\[
1 + \frac{2\theta_+ - 1}{2}(\beta - 1) + \ldots
\]

Hence, as a power series in \( (\beta - 1) \), formula (7.4) is

\[
\alpha = \lambda_1[1 + \frac{1}{2}(\beta - 1) + \ldots], \tag{7.19}
\]

showing that \( \alpha \) is approximately the average of \( \lambda_1 \) and \( \lambda_2 \) when \( \lambda_2 \) is close to \( \lambda_1 \). Quite surprisingly, the next two terms of the Taylor series expansion in (7.19) do not depend on \( \theta_+ \) either, namely,

\[
\alpha = \lambda_1[1 + \frac{1}{2}(\beta - 1) - \frac{1}{6}(\beta - 1)^2 + \frac{1}{12}(\beta - 1)^3 + \ldots]. \tag{7.20}
\]

However, the coefficient of the quartic term does involve \( \theta_+ \). This and (7.20) are easily verified by a mathematical software such as MAPLE or MATHEMATICA. From (7.20) we obtain a simple rule of thumb:
\[ \alpha \approx \frac{\lambda_1 + \lambda_2}{2} - \frac{\lambda_1}{6}(\beta - 1)^2 + \frac{\lambda_1}{12}(\beta - 1)^3. \] (7.21)

Thus knowledge of the exact values of \( \delta_X, \delta_Y \) and \( \delta \) is not needed for this rule.

Furthermore, in a practical situation, \( \lambda_1 \) and \( \lambda_2 \) are not far apart, so that \( \beta - 1 \), the fractional difference between \( \lambda_1 \) and \( \lambda_2 \), is close to 0.

8. Value of a Barrier Strategy for Paying Dividends

The mathematical tools that were developed in the previous sections can be used to study an entirely different problem, the optimal dividend problem. This goes back to an idea of De Finetti (1957) and has been treated in the texts by Borch (1974), Bühlmann (1970), and Gerber (1979), and some of the references cited therein. All these authors analyze the problem in arithmetic growth models for assets and liabilities. Here we consider geometric growth. The asset value, the liability value, and the dividend barrier are modeled by geometric Brownian motions.

For the rest of this paper, consider a company that pays dividends to its shareholders. As a consequence, the original asset value process \( \{A(t)\} \) is replaced by a modified asset value process \( \{A^m(t)\} \). For ease of using the formulas derived earlier, we let \( L_1(t) = L(t) \), the liability value at time \( t \). We assume that if \( A^m(t) \) falls to \( L_1(t) \), the company is “ruined” or becomes bankrupt, and that dividends cannot be paid thereafter. The company is assumed to adopt a barrier strategy for paying dividends. Dividends are paid as an overflow, whenever the modified asset value reaches the barrier \( L_2(t) = \lambda_2 e^{Y(t)} \).

Let \( V(\alpha; \lambda_1, \lambda_2) \) denote the expectation of the discounted dividend payments (until ruin occurs). The quantity \( \alpha - \lambda_1 \) is interpreted as the surplus, equity, net worth or capital of the company at time 0.
We note that \( V(\alpha; \lambda_1, \lambda_2) \) is also a homogeneous function of degree 1. By analogy with (3.6), we have

\[
V(\alpha; \lambda_1, \lambda_2) = J_2(\alpha; \lambda_1, \lambda_2) V(1; \lambda_1/\lambda_2, 1).
\]  

(8.1)

The boundary condition is

\[
V'(\lambda_2; \lambda_1, \lambda_2) = 1,
\]

(8.2)

in analogy to (6.4). From (8.1), (8.2) and (4.6), it follows that

\[
V(\alpha; \lambda_1, \lambda_2) = \lambda_1 \frac{g(\alpha/\lambda_1)}{g(\lambda_2/\lambda_1)}, \quad 0 < \lambda_1 \leq \alpha \leq \lambda_2,
\]

(8.3)

with \( g(z) = z^\theta - z^\delta \) as in (4.7).

9. Optimal Dividends

For given time-0 liability value \( \lambda_1 \) and asset value \( \alpha \), how should \( \lambda_2 \) be chosen? A general idea is to choose \( \lambda_2 \) in order to maximize \( V(\alpha; \lambda_1, \lambda_2) \), the expected present value of all future dividend payments, i.e., to minimize the denominator

\[
g'(\lambda_2/\lambda_1) = \theta_\delta (\lambda_2/\lambda_1)^{\theta_\delta - 1} - \theta_\delta (\lambda_2/\lambda_1)^{\theta - 1}
\]

(9.1)
in (8.3). Note that (9.1) does not depend on \( \alpha \). Four cases have to be distinguished. In the first three cases, the company is “healthy” or “profitable” in the sense that the asset growth rate exceeds the liability growth rate, \( \delta_X > \delta_Y \). Recall that by (2.6), we always assume that the valuation force of interest, \( \delta \), exceeds the liability growth rate, \( \delta_Y \).

(a) If \( 0 < \theta_\delta < 1 \) (\( \delta_X > \delta \)), (9.1) is a decreasing function of \( \lambda_2 \) that tends to 0 for \( \lambda_2 \to \infty \). Hence, the supremum of \( V(\alpha; \lambda_1, \lambda_2) \) is infinite. An interpretation of this result is as follows. Because the asset growth rate is higher than the valuation force of interest,
δ, no earnings are paid out as dividends. It is more profitable that all earnings are reinvested in the company assets.

(b) If \( \theta_+ = 1 \) (\( \delta_X = \delta \)), we get

\[
V(\alpha; \lambda_1, \lambda_2) = \lambda_1 \frac{(\alpha/\lambda_1) - (\alpha/\lambda_1)^{\theta}}{1 - \theta (\lambda_2/\lambda_1)^{\theta-1}}. \tag{9.2}
\]

This is also an increasing function of \( \lambda_2 \), but its limit for \( \lambda_2 \to \infty \) is

\[
\alpha - \alpha^\theta \lambda_1^{1-\theta} \tag{9.3}
\]

and hence finite.

(c) If \( \theta_+ > 1 \) (\( \delta_X < \delta \)), we shall first look at the case where \( \delta_X > \delta_Y \). Let \( \lambda_2^* \) denote the value of \( \lambda_2 \) that maximizes (8.3), i.e., that minimizes (9.1). Setting the derivative of (9.1) with respect to \( \lambda_2 \) equal to zero, we obtain the condition

\[
\theta_+ (\theta_+ - 1) (\lambda_2^*/\lambda_1)^{\theta} = \theta_- (\theta_- - 1) (\lambda_2^*/\lambda_1)^{\theta}, \tag{9.4}
\]

which leads to

\[
\lambda_2^* = \lambda_1 \left[ \frac{\theta_+ (\theta_+ - 1)}{\theta_- (\theta_- - 1)} \right]^{1/\theta - \theta}.
\]

\[
= \lambda_1 \left[ \frac{\delta - \delta_+ - (\delta_X - \delta_Y) \theta_+}{\delta - \delta_- - (\delta_X - \delta_Y) \theta_-} \right]^{1/\theta - \theta} \tag{9.5}
\]

by (5.9). It is clear from the last expression that \( \lambda_2^* > \lambda_1 \).

We now assume the optimal dividend barrier, \( \lambda_2 = \lambda_2^* \), and that the initial asset value is on this barrier, \( \alpha = \lambda_2^* \). Let us evaluate \( V(\lambda_2^*, \lambda_1, \lambda_2^*) \). Formula (8.3) can be rewritten as

\[
V(\alpha; \lambda_1, \lambda_2) = \lambda_2 \frac{(\alpha/\lambda_1)^{\theta} - (\alpha/\lambda_1)^{\theta}}{\theta_+ (\lambda_2/\lambda_1)^{\theta} - \theta_- (\lambda_2/\lambda_1)^{\theta}}. \tag{9.6}
\]
Hence

\[ V(\lambda_2^*; \lambda_1, \lambda_2^*) = \lambda_2^* \frac{(\lambda_2^*/\lambda_1)^\theta - (\lambda_2^*/\lambda_1)^\theta}{\theta_+ (\lambda_2^*/\lambda_1)^\theta - \theta_- (\lambda_2^*/\lambda_1)^\theta} \]

\[ = \lambda_2^* \frac{1}{\theta_+ (\theta_+ - 1)} - \frac{1}{\theta_- (\theta_- - 1)} \]

by (9.4). A simplification yields

\[ V(\lambda_2^*; \lambda_1, \lambda_2^*) = \lambda_2^* \frac{1 - \theta_+ - \theta_-}{\theta_+ \theta_-}. \]  \hspace{1cm} (9.7)

Now, the numerator in (9.7) can be found in (7.18). From (7.17), the denominator is

\[ \theta_+ \theta_- = \frac{-\delta + \delta_Y}{\sigma^2/2}. \]  \hspace{1cm} (9.8)

Applying (7.18) and (9.8) to (9.7) yields the surprisingly simple formula

\[ V(\lambda_2^*; \lambda_1, \lambda_2^*) = \lambda_2^* \frac{\delta_X - \delta_Y}{\delta - \delta_Y}. \]  \hspace{1cm} (9.9)

Furthermore, by the homogeneity property

\[ V(1; \lambda_1/\lambda_2^*, 1) = \frac{\delta_X - \delta_Y}{\delta - \delta_Y}, \]  \hspace{1cm} (9.10)

substitution of which in (8.1) gives

\[ V(\alpha; \lambda_1, \lambda_2^*) = J_2(\alpha; \lambda_1, \lambda_2^*) \frac{\delta_X - \delta_Y}{\delta - \delta_Y}. \]  \hspace{1cm} (9.11)

The investor is interested in the “leverage” ratio,

\[ R(\alpha) = \frac{V(\alpha; \lambda_1, \lambda_2^*)}{\alpha - \lambda_1}, \]  \hspace{1cm} (9.12)
which is the expected present value of all future dividends per dollar invested. Because 
\( V(\lambda_1; \lambda_1, \lambda_2^*) = 0 \), the quantity \( R(\alpha) \) can be interpreted as the slope of a secant. From this 
and the observation that 
\[
V''(\alpha; \lambda_1, \lambda_2^*) < 0 \quad \text{for} \quad \lambda_1 < \alpha < \lambda_2^*,
\]
(9.13) it follows that \( R(\alpha) \) is a decreasing function of \( \alpha \), with 
\[
R(\lambda_1) = V'(\lambda_1; \lambda_1, \lambda_2^*)
= J_2'(\lambda_1; \lambda_1, \lambda_2^*) \frac{\delta_x - \delta_Y}{\delta - \delta_Y}
\]
(9.14) as the maximal value, and 
\[
R(\lambda_2^*) = \frac{\lambda_2^* - \lambda_1}{\lambda_2^* - \lambda_1} \frac{\delta_x - \delta_Y}{\delta - \delta_Y}
\]
(9.15) as the minimal value. Thus, if an investor were only concerned about the leverage ratio, 
he would prefer to invest in companies with a low degree of capitalization.

Furthermore, from (9.13) it follows that the slope of the secant is greater than the 
slope of the tangent at the right end point: for \( \lambda_1 < \alpha < \lambda_2^* \), 
\[
R(\alpha) > V'(\lambda_2^*; \lambda_1, \lambda_2)
= 1
\]
(9.16) by (8.2). In view of (9.12), inequality (9.16) can be restated as 
\[
V(\alpha; \lambda_1, \lambda_2^*) > \alpha - \lambda_1 \quad \text{for} \quad \lambda_1 < \alpha \leq \lambda_2^*.
\]
(9.17) This inequality confirms the fact that, under the conditions \( \delta_Y < \delta \) and \( \delta_Y < \delta_X \), the 
expected present value of all dividends is greater than the initial investment.

(d) Finally, if \( \delta_X \leq \delta_Y \), it is optimal to pay out the amount \( \alpha - \lambda_1 \) immediately as 
dividends and to declare “ruin”.

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10. An Alternative Derivation of (9.9)

We obtained formula (9.9) using brute force. There is another, perhaps more instructive, way to derive it. It follows from (5.2), with $K = V$ and $\alpha = \lambda_2$, and (8.2) that

\[
\frac{\sigma^2}{2} \lambda_2^2 V''(\lambda_2; \lambda_1, \lambda_2) + (\delta_X - \delta_Y) \lambda_2 + (-\delta + \delta_Y) V(\lambda_2; \lambda_1, \lambda_2) = 0. \tag{10.1}
\]

The optimal value $\lambda_2^*$ is determined by the condition

\[
g''(\lambda_2^*/\lambda_1) = 0, \tag{10.2}
\]

applying which to (8.3) yields

\[
V''(\lambda_2^*; \lambda_1, \lambda_2^*) = 0.
\]

Hence we obtain from (10.1)

\[
(\delta_X - \delta_Y) \lambda_2^* + (-\delta + \delta_Y) V(\lambda_2^*; \lambda_1, \lambda_2^*) = 0,
\]

which is equivalent to (9.9).

11. The high contact condition for optimality

Consider again case (c) of Section 9, $0 < \delta_Y < \delta_X < \delta$. Then the optimal value of $\lambda_2$, $\lambda_2^*$, can be characterized by a geometric property, known as the high contact condition in finance literature and the smooth pasting condition in literature about optimal stopping.

The value of a dividend-barrier strategy is given by the function

\[
V(\alpha; \lambda_1, \lambda_2), \quad \text{if } \lambda_1 \leq \alpha \leq \lambda_2;
\]

\[
\alpha - \lambda_2 + V(\lambda_2; \lambda_1, \lambda_2), \quad \text{if } \alpha > \lambda_2.
\]
According to (8.2), this function of $\alpha$ has a continuous first derivative. Under what conditions is the second derivative also a continuous function of $\alpha$, that is,

$$V''(\lambda_2; \lambda_1, \lambda_2) = 0? \quad (11.1)$$

According to (8.3),

$$V''(\lambda_2; \lambda_1, \lambda_2) = \frac{1}{\lambda_1} \cdot \frac{g''(\lambda_2/\lambda_1)}{g'(\lambda_2/\lambda_1)} .$$

Hence, (11.1) holds if and only if $g''(\lambda_2/\lambda_1) = 0$. In view of (10.2), (11.1) holds if and only if $\lambda_2 = \lambda_2^*$. 

12. Concluding Remarks

The expression on the right hand side of formula (9.9) can be obtained as the present value of a perpetuity in a deterministic framework. Suppose that $X(t) = \delta X_t$ and $Y(t) = \delta Y_t$, so that

$$A^m(t) = L_2(t) = \lambda_2 e^{\delta t}, \quad t \geq 0, \quad (12.1)$$

with certainty. Then the “dividend” between time $t$ and time $t+dt$ is

$$(\delta_X - \delta_Y) L_2(t) dt \quad (12.2)$$

and the present value of all dividends is

$$\int_0^\infty e^{-\delta t} (\delta_X - \delta_Y) L_2(t) dt = \lambda_2 \frac{\delta_x - \delta_y}{\delta - \delta_y}. \quad (12.3)$$

The expression on the right-hand side of (12.3), with $\lambda_2$ replaced by (9.5), is identical to the expression on the right-hand side of (9.9). This is somewhat surprising, because in the case of (9.9) the dividends stop at the time of ruin.

In the finance literature, (12.3) is known within the context of the Dividend Discount Model, which is used to determine the price of a stock by discounting its
projected dividend payments. The concept was systematized by John Burr Williams in his 1938 classic treatise. The Dividend Discount Model is also known as the "Gordon model," in honor of Professor Myron J. Gordon who popularized the model; see Gordon (1962).

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References
Appendix

This appendix is to derive the differential equation (5.2) by Itô calculus. Since (5.1) is a martingale, it has zero drift. We can derive (5.2) by finding the drift term in (5.1) and equating it to zero.

Define the standard Brownian motions

\[ W_X(t) = \frac{X(t) - \mu_X t}{\sigma_X} \quad (A.1) \]

and

\[ W_Y(t) = \frac{Y(t) - \mu_Y t}{\sigma_Y} \quad (A.2) \]

Our assumption about the correlation between \{X(t)\} and \{Y(t)\} means that

\[ [dW_X(t)][dW_Y(t)] = \rho dt \quad (A.3) \]

Let

\[ Z(t) = X(t) - Y(t) = (\mu_X - \mu_Y)t + \sigma_X W_X(t) - \sigma_Y W_Y(t) \quad (A.4) \]

and
\[ H = e^{-\delta t + Y(t)} = e^{(-\delta \mu_Y) t + \sigma W_Y(t)}. \]  
(A.5)

Then (5.3) is

\[ e^{-\delta t} K(A(t); L_1(t), L_2(t)) = H K(\alpha e^Z; \lambda_1, \lambda_2). \]  
(A.6)

Now,

\[ d(HK) = (dH)K + H(dK) + (dH)(dK), \]  
(A.7)

\[ dK = K' d(\alpha e^Z) + \frac{1}{2} K'' [d(\alpha e^Z)]^2, \]  
(A.8)

and

\[ de^Z = e^Z [dZ + \frac{1}{2} (dZ)^2] \]

\[ = e^Z [ (\mu_X - \mu_Y) dt + \sigma_X dW_X - \sigma_Y dW_Y + \frac{\sigma^2}{2} dt]. \]  
(A.9)

Recall that \( \sigma \) is defined by (2.8). Applying (A.9) to (A.8) yields

\[ dK = K' \alpha e^Z [(\mu_X - \mu_Y) dt + \sigma_X dW_X - \sigma_Y dW_Y] + K'' (\alpha e^Z)^2 \frac{\sigma^2}{2} dt. \]  
(A.10)

Also, it follows from (A.5) that

\[ dH = H [ (-\delta + \delta_Y) dt + \sigma_Y dW_Y]. \]  
(A.11)

Thus

\[ (dH)(dK) = (H \sigma_Y dW_Y) [K' \alpha e^Z (\sigma_X dW_X - \sigma_Y dW_Y)] \]

\[ = HK' \alpha e^Z (\rho \sigma_Y \sigma_X - \sigma_Y^2) dt. \]  
(A.12)

We now have all the components for finding the coefficient of \( dt \) in (A.7); it is

\[ H [ \frac{\sigma^2}{2} (\alpha e^Z)^2 K'' (\alpha e^Z; \lambda_1, \lambda_2) + (\delta_X - \delta_Y) \alpha e^Z K' (\alpha e^Z; \lambda_1, \lambda_2) + (-\delta + \delta_Y) K (\alpha e^Z; \lambda_1, \lambda_2)]. \]

Equating it to 0, canceling the factor \( H \), and renaming \( \alpha e^Z \) as \( \alpha \), we obtain the differential equation (5.2).
Formula (5.2) can be generalized to the case where the payments are made only up to time $\tau$, $\tau > 0$. Then the time $t$ is also a variable, and we are looking for a function $K(\alpha, t; \lambda_1, \lambda_2)$, with

$$K(\alpha, \tau; \lambda_1, \lambda_2) = 0. \quad (A.13)$$

The process

$$\{e^{-\delta t} K(\lambda(t), t; L_1(t), L_2(t)); 0 \leq t < \min(\tau, T)\} \quad (A.14)$$

is a martingale. It can be shown that the function $K = K(\alpha, t; \lambda_1, \lambda_2)$ satisfies the following partial differential equation:

$$\frac{\partial K}{\partial t} + \frac{\sigma^2}{2} \alpha^2 \frac{\partial^2 K}{\partial \alpha^2} + (\delta X - \delta \gamma) \alpha \frac{\partial K}{\partial \alpha} + (-\delta + \delta \gamma) K = 0. \quad (A.15)$$

Evidently, solving (A.15), with the appropriate boundary conditions, is a more challenging problem than solving (5.2)!