An Efficient Frontier for Participating Policies in a Continuous-time Economy

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Abstract

This paper analyzes trading/payment strategies for participating policies by utilizing the martingale method for optimal portfolio selection problems. We assume that the insurance company continuously invests the premium of the participating policies in the self-financing asset portfolio. The company is also assumed to invest in the portfolio with less risk of the equity return if the expected return of the equity is the same, and invest in the portfolio with a higher expected return if the risk of the equity return is the same. Based on these assumptions, we derive an efficient frontier of the equity return of the company as well as trading strategies to realize efficient portfolios.

1 Introduction

A participating policy is a financial contract that a life insurance company issues to corporate firms. It possesses characteristics such that a minimum interest rate is guaranteed and that bonuses or dividends are paid if the investment performance of the policy is favorable. Thus, from the viewpoint of the policyholders or customers, the payoff of the policy is similar to that of a call option. Here, the underlying asset is a portfolio value held by the company. However, compared to call options, the contract has such distinctive features that the underlying asset depends on the trading strategies

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of the insurance company, the contract continues several years, bonuses are paid every year, if possible, and the precise amount of the each bonus is not known until its payment. These distinctive features certainly affect both the valuation of such policies and trading strategies of the insurance company.

Some recent literatures studying life insurance products as contingent claims are as follows. Brennan and Schwartz (1976) priced unit-linked products referring to Black-Scholes formula (Black and Scholes (1973)). Jensen and Jorgensen (1999) evaluated the path-dependent participating products by the finite difference method, supposing that the asset base evolves according to the geometric Brownian motion. Miltersen and Persson (2000) calculated the combinations of the rate of return guarantees and the fair fraction of the positive excess rate of return which is credited to the customer’s account. They assumed that the rate of return of the specified benchmark portfolio in each year was normally distributed. Giraldi et al. (2000) showed the dynamic hedging strategy for the embedded put option of life insurance products. However, they left the proportions of risky assets of the portfolio, which is the underlying asset of the option, unchanged. Bacinello (2000) indicated the variables controlled by the insurance company are the technical rate, the participation level and riskiness of the reference portfolio. He derived necessary and sufficient conditions under which each control-variable is uniquely determined, given the remaining two.

In this paper, we explicitly consider the dependence of the underlying asset and analyze trading/payment strategies of the company by utilizing the martingale method applied for optimal portfolio selection problems which has been developed in the finance literature (see, e.g., Karatzas and Shreve (1998)). We assume that the company continuously invests the premium of a participating policy in a self-financing asset portfolio according to such a strategy as investing in the portfolio with less risk (or standard deviation) of the equity return if the expected return of the equity is the same, and investing in the portfolio with a higher expected return if the standard deviation of the equity return is the same. Then, similar to the results of the classic mean-variance model for the single-period portfolio selection problem, we show that we can derive an efficient frontier of the equity return in the continuous-time portfolio selection problem. In our model, the efficient frontier has such a property that the expected return of the equity is increasing and concave with respect to the standard deviation of the equity return. So, if the equity-holders of the insurance company are risk-averse, then an optimal asset portfolio is determined as a unique tangent point between the
indifference curve and the efficient frontier, as the modern portfolio theory says. However, we note that although an efficient portfolio on the efficient frontier consists of a risk-less asset and risky assets, the efficient frontier is in general not linear unlike the efficient frontier of the single-period portfolio selection problem. Moreover, we also derive trading strategies to realize the efficient portfolios.

This paper is organized as follows. In the next section, we formally state our model together with the notation necessary for what follows. Section 3 presents our main results. Namely, we derive an efficient frontier of the equity return of the insurance company when the premium of participating policies are invested. We also derive trading strategies to realize the efficient portfolios. In Section 4, we consider the special case that the asset portfolio invested by the company consists of one risk-less asset and one risky asset whose price process of the risky asset follows a geometric Brownian motion. In this case, we can explicitly determine the efficient frontier as well as the trading strategy. A numerical example is given to support the usefulness of our model.

Throughout this paper, all the random variables considered are bounded almost surely (a.s.) to avoid unnecessary technical difficulties. Equalities and inequalities for random variables hold in the sense of a.s.; however, we omit the notation a.s. for the sake of notational simplicity.

2 The Model

We consider a life insurance company that offers a group of participating policies, all maturing at time $T$, $T > 0$, to corporate firms. At time 0, the policyholders pay a premium $\pi_L$ in total to the company to be invested in an asset portfolio. The company contributes an amount of its own capital, equal to $\pi_E$, to acquire the asset portfolio at time 0. Thus, the initial value $\pi_A$ of the portfolio is given by

$$\pi_A = \pi_E + \pi_L.$$  \hfill (1)

We assume that the portfolio is self-financing in the sense that, during time period $\mathcal{T} := [0, T]$, the company will never refund to the policyholders and make no addition and withdrawal of the capital.

The asset portfolio that the company invests consists of a risk-less asset and $n$ risky assets. The price process $\{B(t); t \in \mathcal{T}\}$ of the risk-less asset is
defined by

\[
\frac{dB(t)}{B(t)} = r_f(t)dt, \quad t \in \mathcal{T}, \quad B(0) = 1,
\]

where \( r_f := \{r_f(t); t \in \mathcal{T}\} \) denotes the risk-less interest rate process. On the other hands, an \( n \)-dimensional price process \( \{S(t); t \in \mathcal{T}\} \) of risky assets is defined by

\[
dS(t) = S(t)[\mu(t)dt + \sigma(t)dW(t)], \quad t \in \mathcal{T}, \quad S(0) = s \in (0, \infty)^n.
\]

Here, \( \mu := \{\mu(t); t \in \mathcal{T}\} \) and \( \sigma := \{\sigma(t); t \in \mathcal{T}\} \), respectively, denote the \( n \)-dimensional mean accumulation rate process and the volatility process of the risky assets, \( W := \{W(t); t \in \mathcal{T}\} \) denotes the \( n \)-dimensional standard Brownian motion defined on a given probability space \((\Omega, \mathcal{F}, P)\), and \( S(t) \) denotes an \( n \times n \)-diagonal matrix whose \( i \times i \)-th element is the \( i \)-th element of \( S(t) \). To simplify the following arguments, we assume that \( \sigma \) is an \( n \times n \)-matrix process such that the inverse \( \sigma^{-1}(t) \) of \( \sigma(t) \) exists for each \( t \in \mathcal{T} \). We denote by \( \mathbb{F} := \{\mathcal{F}_t; t \in \mathcal{T}, \mathcal{F}_\mathcal{T} = \mathcal{F}\} \) the \( P \)-augmentation of the natural filtration \( \mathcal{F}_t(t) := \sigma(W(s); s \in [0, t]), t \in \mathcal{T}, \) generated by \( W \). It is assumed that \( r_f, \mu \) and \( \sigma \) are progressively measurable with respect to \( \mathbb{F} \) and satisfy the mild condition

\[
\int_0^T (|r_f(t)| + \|\mu(t)\| + \sigma^2(t))dt < \infty,
\]

where \( \|\cdot\| \) denotes the Euclidian norm and \( \sigma^2(t) \) denotes a sum of the squares of all the elements of \( \sigma(t) \). Throughout the paper, for any probability measure \( P \) on \((\Omega, \mathcal{F})\), we denote the \( \mathcal{F}_t \)-conditioned expectation under \( P \) by \( \mathbb{E}_t^P[\cdot] \) with \( \mathbb{E}_0^P[\cdot] := \mathbb{E}_0^P[\cdot] \). In the case that \( P = P \), we omit the superscript \( P \) for the notational simplicity.

Let \( x := \{x(t) = (x_1(t), \ldots, x_n(t))^\top; t \in \mathcal{T}\} \) be an \( \mathbb{F} \)-adapted stochastic process, where \( x_i(t) \) denotes the number of units invested in the \( i \)-th risky asset at time \( t \), and \( \top \) denotes the transpose. Here and hereafter, all vectors are treated as the column vectors. We refer to \( x \) as a trading strategy. Once a trading strategy \( x \) is given, the value process \( \Pi_A := \{\Pi_A(t); t \in \mathcal{T}\} \) of the asset portfolio is determined so as to follow the stochastic differential equation (abbreviated SDE);

\[
d\Pi_A(t) = r_f(t)\Pi_A(t)dt + x(t)\top S(t)(\mu(t) - r_f(t)\mathbf{1}_n)dt + \sigma(t)dW(t), \quad t \in \mathcal{T},
\]
where $1_n$ denotes the $n$-dimensional unit vector.

We assume that the financial market is frictionless and that there is no arbitrage opportunity. Under these assumptions, provided that $Q$ is an equivalent risk-neutral probability measure on $(\Omega, \mathcal{F})$, we have

$$\frac{\Pi_A(t)}{B(t)} = E^Q_t \left[ \frac{\Pi_A(T)}{B(T)} \right], \quad t \in \mathcal{T}. \quad (5)$$

Here, we note that, if we define the state price density process $\phi := \{\phi(t); t \in \mathcal{T}\}$ by

$$\phi(t) := \exp \left( -\int_0^t \xi(s) \, dW(s) - \int_0^t \left( \frac{1}{2} ||\xi(s)||^2 + r_f(s) \right) \, ds \right), \quad (6)$$

with

$$\xi(t) := \sigma^{-1}(t) (\mu(t) - r_f(t) 1_n), \quad (7)$$

the risk-neutral probability measure $Q$ can be explicitly represented as

$$Q(\mathcal{E}) = E \left[ e^{\int_0^T r_f(t) dt} \phi(T) 1_{\mathcal{E}}, \quad \mathcal{E} \in \mathcal{F}, \quad (8)$$

where

$$1_{\mathcal{E}} = \begin{cases} 1 & \omega \in \mathcal{E} \\ 0 & \omega \notin \mathcal{E} \end{cases}. \quad (9)$$

Furthermore, (5) is rewritten as

$$\Pi_A(t) = B(t)E^Q_t \left[ \frac{\Pi_A(T)}{B(T)} \right] = \frac{1}{\phi(t)} E_t [\phi(T)\Pi_A(T)], \quad t \in \mathcal{T}. \quad (10)$$

We assume that the policies entitle the policyholders to receive the amount $\Pi_L(T)$ in total at the maturity $T$, where

$$\Pi_L(T) = \pi_L \max \left( e^{r_g T}, \ 1 + \beta \frac{\Pi_A(T) - \pi_A}{\pi_A} \right). \quad (11)$$

Here, $r_g$ and $\beta \in (0, 1]$ denote the guaranteed interest rate and the participating rate, respectively, and are positive constants determined at time 0.

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1See, e.g., Karatzas and Shreve (1998) for the definition of the state price density process.
by the contract. Then, the time-$T$ share of the company, $\Pi_E(T)$, among the
value of the asset portfolio becomes the remaining amounts after refunding $\Pi_L(T)$ from $\Pi_A(T)$. That is,

$$\Pi_E(T) = \Pi_A(T) - \Pi_L(T).$$  \hfill (11)

We assume that, for the sake of the equity holders of the company, the
company invests in the asset portfolio according to following strategy.

- Investing in the portfolio with less risk of the equity return if the ex-
  pected return of the equity is the same.
- Investing in the portfolio with a higher expected return if the risk of
  the equity return is the same.

Here and hereafter, risk is assumed to be measured in term of the standard
deviation of the equity return.

In order to derive an optimal trading strategy for the company, we con-
sider the following optimization problem:

$$\begin{align*}
\text{(OP):} & \quad \text{minimize} \quad \mathbb{E} \left[ \left( \Pi_E(T) - \mathbb{E} \left[ \Pi_E(T) \right] \right)^2 \right] \\
& \quad \text{s.to} \quad \mathbb{E} \left[ \phi(T) \Pi_A(T) \right] = \pi_A, \quad \mathbb{E} \left[ \Pi_E(T) \right] = \Pi_E, \quad \hfill (12)
\end{align*}$$

where $\Pi_E$ is a positive constant. In other words, the problem (OP) is to
minimize the variance of the time-$T$ equity value for a given expected value
of the time-$T$ equity, $\Pi_E$, with satisfying the no-arbitrage condition imposed
on the asset portfolio.

If we solve the problem (OP) for various values of $\Pi_E$, we obtain a set
of risk/expectation pairs of the equity return, $(\sigma_R, \bar{R})$, where $\bar{R}$ and $\sigma_R$ are
defined, respectively, by

$$\bar{R} := \mathbb{E}[R]$$

and

$$\sigma_R := \left\{ \mathbb{E}[(R - \bar{R})^2] \right\}^{\frac{1}{2}}$$

with

$$R := \frac{\Pi_E(T) - \pi_E}{\pi_E}.$$

The set of risk/expectation pairs obtained by solving the problem (OP) is
called a \textit{minimum risk boundary} in the feasible risk/expectation universe.
3 Main Results

Before solving the problem \((\text{OP})\), in order to make it more tractable, we represent \(\Pi_A(T)\) and \(\Pi_L(T)\) as functions of \(\Pi_E(T)\). We denote \(\Pi_A(T)\) and \(\Pi_L(T)\), respectively, by \(\Pi_A(\Pi_E(T))\) and \(\Pi_L(\Pi_E(T))\) to emphasize that they are functions of \(\Pi_E(T)\). From (10) and (11), it follows that

\[
\Pi_L(\Pi_E(T)) = \begin{cases} \pi_L e^{\pi_T}; & \Pi_E(T) \leq \Pi_E - K \\ \frac{1}{\beta} [\pi_L (1 - \beta) + (1 - b) \Pi_E(T)]; & \Pi_E(T) > \Pi_E - K \end{cases},
\]

and

\[
\Pi_A(\Pi_E(T)) = \begin{cases} \Pi_E(T) + \pi_L e^{\pi_T}; & \Pi_E(T) \leq \Pi_E - K \\ \frac{1}{\beta} [\Pi_E(T) + \pi_L (1 - \beta)]; & \Pi_E(T) > \Pi_E - K \end{cases},
\]

where

\[
K := \Pi_E - \left(1 + \frac{e^{\pi_T} - 1}{\beta}\right) \pi_A + e^{\pi_T} \pi_L,
\]

and

\[
b := 1 - \beta \frac{\pi_L}{\pi_A}.
\]

Here we note that the inequalities \(\Pi_E(T) > \Pi_E - K\) and \(\Pi_E(T) \leq \Pi_E - K\) represent the condition that the company pays dividends to the policyholders. From (13) and (14), we can rewrite the problem \((\text{OP})\) as

\[
\text{minimize } \mathbb{E} \left[ (\Pi_E(T) - \mathbb{E}[\Pi_E(T)])^2 \right] \text{ s.to } \begin{align*}
\mathbb{E} \left[ \phi(T) \{ (\Pi_E(T) + \pi_L e^{\pi_T}) 1_{\{\Pi_E(T) \leq \Pi_E - K\}} \right. \\
+ \frac{1}{\beta} [\Pi_E(T) + \pi_L (1 - \beta)] 1_{\{\Pi_E(T) > \Pi_E - K\}} \} \right] &= \pi_A, \\
\mathbb{E}[\Pi_E(T)] &= \Pi_E.
\end{align*}
\]

We solve the problem \((\text{OP})\) or \((\text{OP}')\) by the martingale approach (see e.g., Karatzas and Shreve (1998) for details). The next lemma guarantees the existence of a trading strategy for the problem \((\text{OP})\).
Lemma 3.1 For any terminal equity value $\Pi_E(T)$ that satisfies the constraints of the problem (OP), there exists a trading strategy $x$ under which the budget constraint for the asset portfolio,

$$\Pi_A(t) = \frac{1}{\phi(t)} E_t[\phi(T)\Pi_A(T)], \quad t \in T,$$

is satisfied.

Proof. Let $\Pi_A(T)$ be any non-negative random variable satisfying

$$E[\phi(T)\Pi_A(T)] < \infty.$$

We introduce a martingale

$$M(t) := E_t[\phi(T)\Pi_A(T)], \quad t \in T,$$

and invoke the martingale representation theorem of the Brownian filtration to write it as a stochastic integral with respect to the Brownian motion,

$$M(t) = \pi_A + \int_0^t \psi(s)^\top dW(s),$$

for a suitable progressively measurable $\psi : [0, T] \times \Omega \to \mathbb{R}^n$ with $\int_0^T \|\psi(s)\|^2 ds < \infty$. One then identifies

$$\Pi_A(t) = \frac{M(t)}{\phi(t)}, \quad t \in T,$$

as the asset portfolio process corresponding to the trading strategy,

$$x(t) = S^{-1}(t)\sigma^{-1}(t) \left[ \Pi_A(t)\xi(t) + \frac{1}{\phi(t)}\psi(t) \right], \quad t \in T.$$

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Now, corresponding to \((OP')\), we consider the value function

\[
V(\pi_A, \Pi_E) := \inf \left\{ \mathbb{E}\left[ (\Pi_E(T) - \Pi_E)^2 \right]; \phi(T) \left\{ (\Pi_E(T) + \pi_L e^T) 1_{\{\Pi_E(T) \leq \Pi_E - K\}} + \frac{1}{b} [\Pi_E(T) + \pi_L (1 - \beta)] 1_{\{\Pi_E(T) > \Pi_E - K\}} \right\} = \pi_A, \right. \\
\mathbb{E}[\Pi_E(T)] = \Pi_E \right\}. \tag{22}
\]

In order to derive a value of (22), we define a dual function \(\mathcal{D} : \mathbb{R} \times \mathbb{R} \times (0, \infty) \rightarrow \mathbb{R}\) by

\[
\mathcal{D}(a, \ell, \varphi) = \inf_{x \in \mathbb{R}} \mathcal{L}(x; a, \ell, \varphi),
\]

where

\[
\mathcal{L}(x; a, \ell, \varphi) = \left\{ \left[ (x - \Pi_E)^2 - ax - \ell \varphi \left( x + \pi_L e^T \right) \right] 1_{\{x \leq \Pi_E - K\}} \\
+ \left[ (x - \Pi_E)^2 - ax - \ell \varphi \left( \frac{1}{b} x + \frac{1 - \beta}{b} \pi_L \right) \right] 1_{\{x > \Pi_E - K\}} \right\}. \tag{23}
\]

**Lemma 3.2** For each \((a, \ell, \varphi) \in \mathbb{R} \times \mathbb{R} \times (0, \infty)\), we define \(\pi_E^{(+)}(a, \ell, \varphi)\) and
\( \pi_{E}^{(-)}(a, \ell, \varphi) \), respectively, by

\[
\begin{align*}
\pi_{E}^{(+)}(a, \ell, \varphi) &= \left( \Pi_{E} + \frac{a + \ell \varphi}{2} \right) 1_{\frac{a+\ell \varphi}{2}>-\frac{\pi_{E}}{K}}^{\frac{1}{2}a + \frac{1}{2}b\ell\varphi < -\frac{\pi_{E}^{2}}{4}} \\
&= \left( \Pi_{E} + \frac{a + \frac{1}{b} \ell \varphi}{2} \right) 1_{\frac{a+\frac{1}{b} \ell \varphi}{2}>-\frac{\pi_{E}}{K}}^{\frac{1}{2}a + \frac{1}{2}b\ell\varphi < -\frac{\pi_{E}^{2}}{4}},
\end{align*}
\]

Then, if \( \ell > 0 \),

\[
\text{arginf}_{x \in R} \mathcal{L}(x; a, \ell, \varphi) = \pi_{E}^{(+)}(a, \ell, \varphi).
\]

Otherwise,

\[
\text{arginf}_{x \in R} \mathcal{L}(x; a, \ell, \varphi) = \pi_{E}^{(-)}(a, \ell, \varphi).
\]

**Proof.** We make perfect squares of quadratic functions in the right hand side of (23) to obtain

\[
\mathcal{L}(x; a, \ell, \varphi) = \left\{ L_{1}(a, \ell, \varphi)1_{\{x \leq \Pi_{E} - K\}} + L_{2}(a, \ell, \varphi)1_{\{x > \Pi_{E} - K\}} \right\},
\]

where

\[
L_{1}(a, \ell, \varphi) := \left( x - \Pi_{E} - \frac{1}{2}(a + \ell \varphi) \right)^{2} + C_{1}(a, \ell \varphi),
\]

and

\[
L_{2}(a, \ell, \varphi) := \left( x - \Pi_{E} - \frac{1}{2} \left( a + \frac{1}{b} \ell \varphi \right) \right)^{2} + C_{2}(a, \ell \varphi),
\]

with

\[
C_{1}(a, \ell, \varphi) := -(a + \ell \varphi)\Pi_{E} - \left( \frac{a + \ell \varphi}{2} \right)^{2} - \ell \varphi \pi_{L} e^{r_{1}T}.
\]
and
\[
C_2(a, \ell, \varphi) := C_1(a, \ell, \varphi) + \left(1 - \frac{1}{b}\right) \left\{ \ell K + \frac{\ell}{2} \left( a + \ell \varphi \right) + \frac{\ell}{2} \left( a + \frac{1}{b} \ell \varphi \right) \right\} \varphi.
\]

This leads to the desired result (see Figure 1). \(\square\)

Lemma 3.2 leads to our main result which states a solution of the problem (OP).

**Theorem 3.1** Let \((a^*, \ell^*)\) be a solution of the simultaneous equation of \((a, \ell)\) given by
\[
\begin{align*}
\left\{ \begin{array}{l}
E[\pi_E^{(+)}(a, \ell, \phi(T))] = \Pi_E, \\
E[\phi(T)\Pi_A(\pi_E^{(+)}(a, \ell, \phi(T)))] = \pi_A.
\end{array} \right.
\end{align*}
\]

(24)

Then, \(\ell^* > 0\), \(a^* < 0\), and the value function \(V(\pi_A, \Pi_E)\) and the corresponding time-\(T\) share of the company \(\Pi_E^*(T)\) are, respectively, given by
\[
V(\pi_A, \Pi_E) = E \left[ \left( \frac{a^* + \ell^* \phi(T)}{2} \right)^2 1_{\frac{2a^* + 1}{4} \ell^* \phi(T) < -K} \right],
\]
\[
+ \left( \frac{a^* + \frac{1}{b} \ell^* \phi(T)}{2} \right)^2 1_{\frac{2a^* + 1}{4} \ell^* \phi(T) \geq -K} \right],
\]

(25)

and
\[
\Pi_E^*(T) = \pi_E^{(+)}(a^*, \ell^*, \phi(T)).
\]

(26)

On the other hand, let \((a^*, \ell^*)\) be a solution of the simultaneous equation of \((a, \ell)\) given by
\[
\begin{align*}
\left\{ \begin{array}{l}
E[\pi_E^{(-)}(a, \ell, \phi(T))] = \Pi_E, \\
E[\phi(T)\Pi_A(\pi_E^{(-)}(a, \ell, \phi(T)))] = \pi_A.
\end{array} \right.
\end{align*}
\]

(27)
Then, \( \ell^* < 0, \ a^* > 0 \), and the value function \( V(\pi_A, \Pi_E) \) and the corresponding time-\( T \) share of the company \( \Pi^*_E(T) \) are, respectively, given by

\[
V(\pi_A, \Pi_E) = \mathbb{E} \left[ \left( \frac{a^* + \ell^* \phi(T)}{2} \right)^2 1\{\frac{a^* + \ell^* \phi(T)}{2} \leq -K\} ight] + K^2 1\{a^* + \ell^* \phi(T) \leq -K\} \prod_E^E(T) \leq -K \leq \frac{a^* + \ell^* \phi(T)}{2} \}
\]

and

\[
\Pi^*_E(T) = \pi^E_-(a^*, \ell^*, \phi(T)).
\]

**Proof.** From Lemma 3.2, we can easily confirm that (25), (26), (28), and (29) hold. Thus, we have only to show that if \( \ell^* > 0 \), then \( a^* < 0 \); otherwise, \( a^* > 0 \). Since \( \Pi^*_E(T) \) is given by (26) for the case that \( \ell > 0 \), \( \pi^E_+(a^*, \ell^*, \phi(T)) \) must satisfy the second constraint of (OP), which is equivalent to the first equation of (24). This leads to the conclusion. The other case can be shown similarly.

From Lemma 3.1, we obtain the following corollary, which says about the trading strategy to attain a solution to the problem (OP).

**Corollary 3.1** A trading strategy to attain the solution to the problem (OP) is given by (17) and (21) with \( \Pi_A(T) = \Pi_A(\Pi^*_E(T)) \).

When \( \ell^* < 0 \), we refer the minimum risk boundary in the risk/expectation universe \( (\sigma_R, \Pi) \) as an efficient frontier. The reason why we call it the efficient frontier is partially shown by the next theorem. It says about the shape of the efficient frontier.

**Theorem 3.2** On the minimum risk boundary, if \( \ell^* < 0 \), the expected return of the equity is a strictly increasing and strictly concave function of the risk; otherwise, it is a strictly decreasing and concave function of the risk. That is, let

\[
\mathcal{R}^* := \mathbb{E} \left[ \frac{\Pi^*_E(T)}{\pi_E} - 1 \right] = \frac{\Pi_E}{\pi_E} - 1
\]
and

\[ \sigma^*_R := \left\{ \mathbb{E} \left[ \left( \frac{\Pi^E_{E}(T) - \Pi_E}{\pi_E} \right)^2 \right] \right\}^{\frac{1}{2}} = \sqrt{\frac{V(\pi_A, \Pi_E)}{\pi_E}}. \quad (31) \]

Then, if \( \ell^* < 0 \),

\[ \frac{dR^*}{d\sigma^*_R} > 0, \quad \frac{d^2R^*}{(d\sigma^*_R)^2} < 0, \]

otherwise,

\[ \frac{dR^*}{d\sigma^*_R} < 0, \quad \frac{d^2R^*}{(d\sigma^*_R)^2} \leq 0. \]

**Proof.** We only prove the case that \( \ell^* < 0 \) since the other case can be proven quite similarly. We first show that \( \frac{\partial V(\pi_A, \Pi_E)}{\partial \Pi_E} = a^* \). If so, from (30) and (31), we then have \( \frac{\sigma^*_R}{R^*} = \frac{1}{2\sqrt{V(\pi_A, \Pi_E)}}a^* \), since \( \ell^* < 0 \) is assumed. Now, corresponding to (27), we define the implicit functions \( f(a, \ell, \Pi_E) = 0 \) and \( g(a, \ell, \Pi_E) = 0 \), respectively, by

\[ f(a, \ell, \Pi_E) = \mathbb{E}[\pi_{E}^{(-)}(a, \ell, \phi(T))] - \Pi_E \]

and

\[ g(a, \ell, \Pi_E) = \mathbb{E}[\phi(T)\Pi_{A}(\pi_{E}^{(-)}(a, \ell, \phi(T))))] - \pi_A. \]

Then, a tedious algebra shows that

\[ \left( \begin{array}{c} \frac{da}{d\Pi_E} \\ \frac{da}{d\ell} \end{array} \right) = \left( \begin{array}{cc} \frac{\partial f}{\partial a} & \frac{\partial f}{\partial \ell} \\ \frac{\partial g}{\partial a} & \frac{\partial g}{\partial \ell} \end{array} \right)^{-1} \left( \begin{array}{c} \frac{\partial f}{\partial \Pi_E} \\ \frac{\partial g}{\partial \Pi_E} \end{array} \right) = -\frac{1}{2D} \left( \begin{array}{c} 4D - \Psi_3 \\ \Psi_2 \end{array} \right), \quad (32) \]

where

\[ D := \frac{1}{4}(\Psi_1\Psi_3 - \Psi_2^2), \]

\[ \Psi_1 := \mathbb{E}[1_A + 1_B], \quad \Psi_2 := \mathbb{E}\left[\phi(T)\left(\frac{1}{B}1_A + 1_B\right)\right], \]

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\[
\Psi_3 := \mathbb{E} \left[ \phi(T)^2 \left( \frac{1}{b^2} 1_A + 1_B \right) \right],
\]

with
\[
A := \left\{ \omega \in \Omega; \ K + \frac{a + \ell \phi(T)}{2} > 0 \right\},
\]
\[
B := \left\{ \omega \in \Omega; \ K + \frac{a + \ell \phi(T)}{2} \leq 0 \right\}.
\]

From (32) and (28), we have \( \frac{\partial V(\pi_A, \Pi_E)}{\partial \Pi_E} = a^* \), as claimed.

Next, we show that \( \frac{\partial^2 R}{\partial \sigma_R^2} < 0 \). It can be readily seen that
\[
\frac{d^2 R}{d(\sigma_R^2)} = \frac{2\pi_E}{a^*^3} \left( a^*^2 - 2V(\pi_A, \Pi_E) \frac{da^*}{d\Pi_E} \right).
\]

Also, (27) is equivalent to the equation
\[
\begin{pmatrix}
\Psi_1 - 1 & \Psi_2 \\
\Psi_2 & \Psi_3
\end{pmatrix}
\begin{pmatrix}
K \\
\ell/2
\end{pmatrix}
= \begin{pmatrix}
-\frac{a}{2} \Psi_1 \\
-\frac{a}{2} \Psi_2 + \kappa
\end{pmatrix},
\]

where
\[
\kappa := \left( 1 - \left( 1 + \frac{e^{\gamma T} - 1}{\beta} \right) \mathbb{E}[\phi(T)] \right) \pi_A.
\]

Hence, we obtain
\[
\begin{pmatrix}
\ell/2 \\
\ell/2
\end{pmatrix}
= \frac{1}{4D - \Psi_3} \begin{pmatrix}
-2aD - \kappa \Psi_2 \\
\frac{a}{2} \Psi_2 - \kappa (1 - \Psi_1)
\end{pmatrix}.
\]

Substituting this into (28) leads that
\[
a^*^2 - 2V(\pi_A, \Pi_E) \frac{da^*}{d\Pi_E} = -\frac{\kappa^2}{D} (1 - \Psi_1).
\]
Furthermore, since

\[ 4D = \mathbb{E} \left[ \phi(T)^2 \left( \frac{1}{b^2} 1_A + 1_B \right) \right] \mathbb{E} [1_A + 1_B] - \mathbb{E} \left[ \phi(T) \left( \frac{1}{b} 1_A + 1_B \right) \right]^2 \]

\[ = \mathbb{E} \left[ \left( \phi(T) \left( \frac{1}{b} 1_A + 1_B \right) \right)^2 \right] \mathbb{E} [(1_A + 1_B)^2] \]

\[ - \mathbb{E} \left[ (1_A + 1_B) \phi(T) \left( \frac{1}{b} 1_A + 1_B \right) \right]^2 , \]

we have \( D > 0 \) by the Cauchy-Schwartz inequality. Hence, from (33), we can conclude that \( \frac{d^2 \mathbb{R}}{(d\sigma_R)^2} < 0. \)

From Theorem 3.2, if an indifference curve of the equity-holders on \((\sigma_R, \mathbb{R})\)-plain is such that \( \mathbb{R} \) is an increasing and convex function of \( \sigma_R \), i.e., the equity-holders are risk-averse, then an optimal asset portfolio is determined as a unique tangent point between the indifference curve and the efficient frontier. However, we note that, although an efficient portfolio on the efficient frontier consists of a risk-less asset and \( n \)-risky assets, the efficient frontier is in general not linear unlike the single-period portfolio selection problem.

### 4 A Special Case: The Geometric Brownian Motion

In this section, we consider the special case that the number of risky assets \( n \) is equal to 1, that the process of the risky asset price follows a geometric Brownian motion, and that the interest rate is constant. Namely, suppose that, for all \( t \in \mathcal{T} \), \( \mu(t) \), \( \sigma(t) \) and \( r_f(t) \) are positive constants, \( \mu \), \( \sigma \) and \( r_f \) say, respectively, in (2) and (3). Then, \( \xi(t) \) is also a constant, \( \xi \) say, and \( \ln \phi(t) \) obeys a normal distribution with mean \( (-r_f - \frac{1}{2} \xi^2) t \) and standard deviation \( \xi \sqrt{t} \). This section explicitly derives an efficient frontier of equity return and trading strategies to attain efficient portfolios.
From (28) and (31), we have

\[
\pi_E^2 \sigma_R^2 = E \left[ \left( \Pi_E^\ell (a, \ell, \phi(T)) - \Pi_E \right)^2 \right] \tag{34}
\]

\[
= \frac{a^2}{4} \left( 1 - \Phi \left( d_2(0) - \xi \sqrt{T} \right) \right) + \frac{a \ell}{2b} \exp (-r_f T) (1 - \Phi(d_2(0))) + \frac{\ell^2}{4b^2} \exp \left( (\xi^2 - 2r_f)T \right) \left( 1 - \Phi \left( d_2(0) + \xi \sqrt{T} \right) \right) + K^2 \left( \Phi \left( d_2(0) - \xi \sqrt{T} \right) - \Phi \left( d_1(0) - \xi \sqrt{T} \right) \right) + \frac{a^2}{4} \Phi \left( d_1(0) - \xi \sqrt{T} \right) + \frac{a \ell}{2} \exp (-r_f T) \Phi \left( d_1(0) \right) + \frac{\ell^2}{4} \exp \left( (\xi^2 - 2r_f)T \right) \Phi \left( d_1(0) + \xi \sqrt{T} \right), \tag{35}
\]

where

\[
d_1(t) := \ln \left( \frac{\ell \phi(t)}{-2K - a} \right) + \frac{1}{2} \xi^2 - r_f \right) \frac{(T - t)}{\xi \sqrt{T - t}},
\]

\[
d_2(t) := \ln \left( \frac{\ell \phi(t)}{b(-2K - a)} \right) + \frac{1}{2} \xi^2 - r_f \right) \frac{(T - t)}{\xi \sqrt{T - t}},
\]

and \( \Phi(\cdot) \) denotes the distribution function of the standard normal distribution. Thus, calculating \( a \) and \( \ell \) for each value of \( \Pi_E \), we can obtain an efficient frontier on a risk/expectation plane of the equity return \( \dot{R} \). In order to obtain the values of \( a^* \) and \( \ell^* \), we have only to solve (27). Figure 2 depicts the efficient frontier for the case that \( T = 1, \ r_f = 0.05, \ r_g = 0.03, \ B(0) = 1, \ S(0) = 1, \ \mu = 0.1, \ \sigma = 0.3, \ \beta = 0.9, \ \pi_A = 1.2, \ \text{and} \ \pi_L = 1.0. \)

Finally, we derive the trading strategy to attain the efficient portfolio which represented by \( (\dot{R}^*, \sigma_{\dot{R}}^*) \). Noting that the value \( \Pi_A(T) \) of the efficient portfolio is given by \( \Pi_A(\Pi_E^\ell (T)) \), and applying Ito’s formula to \( \Pi_A(t) \equiv \mathbb{E}^Q_t [\Pi_A(\Pi_E(T))] \), we obtain

\[
d\Pi_A(t) = r_f \Pi_A(t) dt + e^{-r_f(T-t)} \mathcal{Y}(t) (dW(t) + \xi dt), \tag{36}
\]
where

\[ \mathcal{Y}(t) := -\frac{\xi\xi}{2} \phi(t) e^{(\xi^2 - r_f)(T-t)} \times \left( \frac{1}{\sqrt{2\pi}} \left( 1 - \Phi \left( d_2(t) + \xi \sqrt{T-t} \right) \right) + \Phi \left( d_1(t) + \xi \sqrt{T-t} \right) \right). \]

Comparing the coefficient of \( dW(t) \) in (36) with that in (4), we conclude that the trading strategy \( \{ x(t) \equiv x^*(t); t \in T \} \) to realize \( \Pi_A(\Pi_E(T)) \) is given by

\[ x^*(t) = \frac{e^{-r_f(T-t)} \mathcal{Y}(t)}{S(t)\sigma}, \quad t \in T. \tag{37} \]

5 Concluding Remarks

In this paper we have analyzed trading/payment strategies for participating policies. We studied the policy as the contingent claim whose underlying asset is an asset portfolio that an insurance company is able to control continuously. We derived an efficient frontier of the equity return of the company as well as trading strategies to realize efficient portfolios. We found that the efficient frontier is a convex and increasing function of the standard deviation of the equity return unlike the single-period portfolio selection problem, although an efficient portfolio on the efficient frontier consists of a risk-less asset and \( n \)-risky assets.

Following problems are left for further research. Firstly we should study some comparative statics for, e.g., the analytic characteristics of the optimal portfolio. Secondarily more realistic models including such as multiple bonus payment property for life insurance products should be examined.
\( \ell \geq 0; \)

\[
-K < \frac{a + \ell \varphi}{2} \quad \frac{a + \ell \varphi}{2} \leq -K < \frac{a + \frac{1}{2} \ell \varphi}{2} \quad \frac{a + \frac{1}{2} \ell \varphi}{2} \leq -K
\]

\[
\text{Figure 1: Graphs of } \mathcal{L}(x; a, \ell, \varphi)
\]

\( \ell < 0; \)

\[
-K < \frac{a + \frac{1}{2} \ell \varphi}{2} \quad \frac{a + \frac{1}{2} \ell \varphi}{2} \leq -K < \frac{a + \ell \varphi}{2} \quad \frac{a + \ell \varphi}{2} \leq -K
\]
References


