Distortion Risk Measures: Coherence and Stochastic Dominance

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Abstract

In this paper it is proved that a concave distortion function is a necessary and sufficient condition for coherence, and a strictly concave distortion function is a necessary and sufficient condition for strict consistency with second order stochastic dominance. The results are related to current risk measures used in practice, such as value-at-risk (VaR) and the conditional tail expectation (CTE), also known as tail-VaR and to Wang's premium principles.

Key Words: Risk Measure, Coherence, Stochastic Dominance, Distortion

1 Introduction

1.1 Outline

Distortion risk measures form an important class; they include Value at Risk, Conditional Tail Expectation and Wang's PH transform premium principle. Later in this section we review distortion functions and distortion risk measures.

The first objective of this paper is to lay out the relationship between the characteristics of these risk measures and the criteria for *coherence* proposed by Artzner *et al* (1999), and in section 2 we show that a distortion risk measure is coherent if and only if the associated distortion function is concave.

The second objective is to consider a stronger requirement, that random variables that can be ordered using second order stochastic dominance should generate risk measures that retain that ordering. That is, if X is preferred to Y under second order stochastic dominance then the risk measure for X should be less than the risk measure for Y. In Section 3 we show that a necessary and sufficient condition for this ordering to follow strictly using distortion risk measures, is that the distortion function should be strictly concave.

1.2 Coherent capital requirements

Given a loss random variable X, the risk measure is a functional $\rho(X) : X \mapsto [0, \infty)$. The premium principle is the most commonly recognized actuarial risk measure, but principles for capital requirements are also coming into recognition. Capital requirement risk measures are used to determine the capital required in respect of a random loss X with a view to avoiding insolvency. The most common capital requirement risk measure in common use is the Value at Risk, or VaR measure. This is better known by actuaries as a quantile reserving principle, where the reserve requirement is a quantile of the loss distribution.

Many of the accepted requisites for premium risk measures also apply to capital requirement risk measures. One of the major differences lies in recognition of gains. For a risk which may result in a gain ($X \leq 0$) or a loss ($X > 0$), it is appropriate in pricing the risk to take consideration of the potential gains. In capital adequacy this can lead to unacceptable results – for example, a negative capital requirement. For the purpose of ensuring solvency in the event of adverse experience, the gain side of the distribution should not be allowed to offset the losses. For results consistent with the objectives of captal requirement calculations, it is therefore appropriate to use a loss distribution censored at zero, and this censoring is assumed in this paper.

In Artzner *et al* (1999), a set of four axioms for a 'coherent' risk measure for capital adequacy are proposed. It is then demonstrated that a quantile risk measure, such as Value-at-Risk, or VaR, does not satisfy these axioms, but that a measure based on the expectation in the right tail of the loss distribution, (Conditional Tail Expectation, or Tail-VaR) does satisfy the axioms and is therefore preferable.

The coherency axioms for a risk measure are:

$$\text{Bounded above by the maximum loss:} \quad \rho(X) \leq \max(X) \tag{1}$$

$$\text{Bounded below by the mean loss:} \quad \rho(X) \geq \mathrm{E}[X] \tag{2}$$

$$\text{Scalar additive and multiplicative:} \quad \rho(aX+b) = a\rho(X)+b, \;\text{ for } a,b>0 \tag{3}$$

$$\text{Subadditive:} \quad \rho(X+Y) \leq \rho(X)+\rho(Y) \tag{4}$$

1.3 Distortion risk measures

Distortion risk measures developed from research on premium principles by Wang (1995), and are defined as follows:

A distortion function is a non-decreasing function with $g(0) = 0$ and $g(1) = 1$, and $g : [0,1] \rightarrow [0,1]$.

A distortion risk measure associated with distortion function g, for a random loss X with decumulative distribution function $S(x)$ is

$$\rho_g(X) = \int_0^\infty g(S(x))dx \tag{5}$$

The distortion risk measure adjusts the true probability measure to give more weight to higher risk events. The function $g(S(x))$ can be thought of as a risk adjusted decumulative distribution function. Since X is a non-negative random variable, $\rho(X) \equiv E_g[X]$ where the subscript indicates the change of measure.

Using distorted probabilities, it is possible to define a distortion $g_V()$ that will produce the traditional VaR measure, V_α, as the risk measure.

$$g_V(t) = \begin{cases} 1 & if \quad 1-\alpha < t \leq 1, \\ 0 & if \quad 0 < t < 1-\alpha. \end{cases} \tag{6}$$

so that the risk measure is

$$\rho_v(X) = \int_0^\infty g_v(S(x))dx = \int_0^{V_\alpha} dx = V_\alpha \tag{7}$$

where V_α is $F_X^{-1}(\alpha)$

The conditional tail expectation (CTE) or Tail-VaR is defined for smooth distribution functions, given the parameter α, $0 < \alpha < 1$, as:

$$CTE_\alpha = E[\ X \mid X > F_X^{-1}(\alpha)\ ]. \tag{8}$$

where $F_X^{-1}()$ is the inverse distribution function of the loss random variable, X. That is, $F_X^{-1}(\alpha)$ is the 100α percentile of the loss distribution.

The CTE can also be expressed in terms of a distortion risk measure as follows:

$$g_c(t) = \begin{cases} 1 & if \quad 1-\alpha < t \leq 1, \\ \frac{t}{1-\alpha} & if \quad 0 < t < 1-\alpha. \end{cases} \tag{9}$$

Both of these measures use only the tail of the distribution. Using the work of Wang (1995, 1996) and Wang, Young and Panjer (1997) on premium principles, it can be seen that there are advantages to distortions which utilize the whole censored loss distribution. The Beta-distortion risk measure uses the incomplete beta function:

$$g(S(x)) = \beta(a,b;S(x)) = \int_0^{S(x)} \frac{1}{\beta(a,b)} t^{a-1}(1-t)^{b-1}dt = F_\beta(S(x)) \tag{10}$$

where $F_\beta(x)$ is the distribution function of the beta distribution, and $\beta(a,b)$ is the beta function with parameters $a > 0$ and $b > 0$; that is:

$$\beta(a,b) = \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)} = \int_0^1 t^{a-1}(1-t)^{b-1}dt \tag{11}$$

The Beta-distortion risk measure is concave if and only if $a \le 1$ and $b \ge 1$, and is strictly concave if a and b are not both equal to 1 (see Wirch (1999b)). The PH-transform is a special case of the Beta-distortion risk measure. The PH-transform risk measure is defined as:

$$\rho_{PH}(X) = \int_0^\infty S_X(x)^{\frac{1}{\gamma}} dx, \quad \gamma > 1 \tag{12}$$

2 Coherence and concave distortions

All distortion functions generate distortion risk measures which satisfy the first and third criteria of coherence, (1) and (3) above.

In this section, in the following two theorems we show necessary and sufficient conditions for the expected value and subadditive criteria, (2) and (4) above.

Theorem 2.1 *The distortion risk measure, $\rho_g(X) = \int_0^\infty g(S_X(x))dx$ is bounded below by the mean loss if and only if $g(t) \ge t$ for $t \in [0,1]$.*

Proof: The condition is obviously sufficient as

$$\rho_g(X) = \int_0^\infty g(S_X(x))dx \ge \int_0^\infty S_X(x)dx = E[X] \tag{13}$$

We use a contradiction argument to show that the condition is also necessary. Assume for t in some region $0 \le a < b \le 1$, $g(t) < t$. For any z such that $a < z < b$ we can construct the random variable

$$X = \begin{cases} 0 & \text{with probability} \quad 1-z \\ w > 0 & \text{with probability} \quad z \end{cases}$$

Then E[X]=zw and $\rho(X) = wg(z) < wz < E[X]$.

For the final condition, subadditivity, we show that it is necessary and sufficient that the distortion function is concave. As $g(0) = 0$ and $g(1) = 1$, concavity implies that for any $t \in [0,1]$ $g(t) \ge t$, so this is a stronger condition than that used in Theorem 2.1.

Theorem 2.2 *The distortion risk measure, $\rho_g(X) = \int_0^\infty g(S_X(x))dx$ is subadditive if and only if g is a concave distortion function.*

Proof: That a concave distortion function is sufficient for sub-additivity is stated but not proved in Wang (1996). The result is proved in Wang and Dhaene (1998) using the property that concave distortion risk measures are additive for comonotonic risks.

We show that concavity is a necessary condition by showing that a distortion function with a strictly convex section cannot be subadditive.

Suppose that $g(t)$ is strictly convex in the region $0 \leq a < t < b \leq 1$. Let $c = (a+b)/2$.

Convexity in this region implies that for any $z < (b-a)/2$

$$g(c+z) - g(c) > g(c) - g(c-z) \tag{14}$$

Consider the random variables X and Y with discrete joint distribution for any $w > 0$ and for any $z < (b-a)/2$:

		Y: 0	$w+z/2$	$w+z$	Σ
X	0	$1-c-z$	z	0	$1-c$
	$w+z$	z	0	$c-z$	c
	Σ	$1-c$	z	$c-z$	1

For these random variables

$$\begin{aligned} \rho(X) &= (w+z)g(c) && (15)\\ \rho(Y) &= (w+\frac{z}{2})g(c) + \frac{z}{2}\,g(c-z) && (16)\\ \rho(X+Y) &= (w+\frac{z}{2})g(c+z) + \frac{z}{2}g(c) + (w+z)g(c-z) && (17) \end{aligned}$$

So that

$$\begin{aligned} &\rho(X+Y) - (\rho(X)+\rho(Y)) \\ &= \left(w+\frac{z}{2}\right)\Big\{ (g(c+z)-g(c)) - (g(c)-g(c-z)) \Big\} \qquad (18)\\ &> 0 \qquad \text{from (14)} \end{aligned}$$

Both the CTE and beta distortion functions are concave, and therefore coherent. The VaR distortion is not concave, and does not satisfy the criterion that $g(t) > t$, so VaR fails to be coherent because it is not a subadditive measure and because it is not bounded below by the mean value of the random variable.

Because the concavity criterion of Theorem 2.2 is stronger than the $g(t) \geq t$ criterion of Theorem 2.1 concavity is a necessary and sufficient condition for coherence for distortion risk functions.

3 Stochastic Order

In Section 2 concavity of $g(t)$ was shown to be an important feature. In this section we explore more deeply the difference between strictly concave distortion functions ($g''_\beta(t) < 0$ for all t) and non-strictly concave distortions ($g''_\beta(t) = 0$ for some t). The beta distortion function is strictly concave, whereas the piecewise linear distortion functions, including the CTE, are not.

Consider losses X and Y. Let $\succ$ indicate preference under partial ordering, so that $X \succ Y$ means that X is preferred to Y

We say that a risk measure $\rho()$ preserves a stochastic ordering $\succ$ if

$$X \succ Y \Longrightarrow \rho(X) \leq \rho(Y).$$

The ordering is strongly preserved if strict stochastic ordering of random variables leads to strict ordering of the risk measure.

First Order Stochastic Dominance If $S_X(t) \leq S_Y(t)$ for all $t \geq 0$, and $S_X(t) < S_Y(t)$ for some $t \geq 0$, then $X \succ_{1st} Y$.
(Note: There are many other equivalent conditions. See Wang (1998).)

All distortion functions preserve first order stochastic dominance. That is, if $X \succ_{1st} Y$ then $\rho(X) \leq \rho(Y)$ follows from the fact that the distortion function is an increasing function.

Second Order Stochastic Dominance : For any two risks X and Y, if

$$\int_x^\infty S_X(t)dt \leq \int_x^\infty S_Y(t)dt,$$

for all $x \geq 0$, with strict inequality for some $x \in (0, \infty)$ then we say that X succedes Y in second order stochastic dominance, or $X \succ_{2nd} Y$.
(Note: There are many other equivalent conditions such as stop-loss order. See Wang(1998))

Not all risks can be ordered using second order stochastic dominance. Any pair of risks with survival distributions which cross an even number of times, cannot be compared, as the sign of the difference in integrals before the first crossing and after the last crossing are opposite.

Where we can order random variables with second order stochastic dominance, an additional property that would be attractive in a risk measure is that it strongly preserves second order stochastic dominance. In fact, as we show below, this depends on whether or not the distortion function is strictly concave.

Theorem 3.1 *For a risk measure $\rho(X) = \int_0^\infty g(S(x))dx$ where $g()$ is strictly concave, then $X \succ_{2nd} Y \Rightarrow \rho(X) < \rho(Y)$*

Proof: (based on proof from Wang (1996))

Due to Müller (1996) we only have to prove that the increasing, strictly concave distortion risk measures preserve second order stochastic dominance where the decumulative distribution functions cross once only.

Let $E[X] \leq E[Y]$, $X \succ_{2nd} Y$ and let t_0 be the once crossing point, so that

$$\begin{array}{ll} S_X(t) \geq S_Y(t) & \text{for } t < t_0 \\ S_X(t) \leq S_Y(t) & \text{for } t \geq t_0 \end{array}$$

and since $X \succ_{2nd} Y$ either

$$\begin{array}{rcll} S_X(t) & < & S_Y(t) & \text{for some } \; t > t_0 \\ \text{and/or } S_X(t) & > & S_Y(t) & \text{for some } \; t < t_0 \end{array}$$

Next, construct a new ddf,

$$S_Z(t) = \max\{S_X(t),\ S_Y(t)\} = \begin{cases} S_X(t) & t < t_0 \\ S_Y(t) & t \geq t_0 \end{cases}$$

so that:

$$\rho_g(Z) - \rho_g(X) = \int_{t_0}^{\infty} [g(S_Y(t)) - g(S_X(t))]dt \tag{19}$$

for $t > t_0$, $\quad S(t_0) \geq S_Y(t) \geq S_X(t)$
If $S_Y(t) = S_X(t)$ then $\rho_g(Z) - \rho_g(X) = 0$,
So let $\quad S(t_0) > S_Y(t) > S_X(t)$ for some $t > t_0$.
For some $\epsilon > 0$,

$$\frac{g(S_Y(t)) - g(S_X(t)) - \epsilon}{S_Y(t) - S_X(t)} = \frac{g(S(t_0)) - g(S_X(t))}{S(t_0) - S_X(t)}$$

which implies

$$
\begin{aligned}
g(S_Y(t)) - g(S_X(t)) &> (S_Y(t) - S_X(t))\frac{g(S(t_0)) - g(S_X(t))}{S(t_0) - S_X(t)} \\
&> (S_Y(t) - S_X(t))g'(S(t_0))
\end{aligned}
$$

So for $t > t_0$,

$$\rho_g(Z) - \rho_g(X) \geq g'(S(t_0)) \int_{t_0}^{\infty} S_Y(t) - S_X(t)dt \tag{20}$$

and

$$\rho_g(Z) - \rho_g(Y) = \int_0^{t_0} [g(S_X(t)) - g(S_Y(t))]dt \tag{21}$$

for $t < t_0, \quad S_X(t) \geq S_Y(t) \geq S(t_0)$
If $S_Y(t) = S_X(t)$ then $\rho_g(Z) - \rho_g(Y) = 0$,
So let $\quad S_X(t) > S_Y(t) > S(t_0)$ for some $t < t_0$.
For some $\epsilon > 0$,

$$\frac{g(S_X(t)) - g(S_Y(t)) + \epsilon}{S_X(t) - S_Y(t)} = \frac{g(S_X(t)) - g(S(t_0))}{S_X(t) - S(t_0)}$$

which implies

$$
\begin{aligned}
g(S_X(t)) - g(S_Y(t)) &< (S_X(t) - S_Y(t))\frac{g(S_X(t)) - g(S(t_0))}{S_X(t) - S(t_0)} \\
&< (S_X(t) - S_Y(t))g'(S(t_0))
\end{aligned}
$$

So for $t < t_0$,

$$\rho_g(Z) - \rho_g(Y) \geq g'(S(t_0)) \int_0^{t_0} [S_X(t) - S_Y(t)]dt \tag{22}$$

with at least one of the above inequalities, (20) or (22), being a strict inequality.
Subtracting the last two equations, we obtain

$$\rho_g(Y) - \rho_g(X) > g'(S(t_0)) \int_0^{\infty} [S_Y(t) - S_X(t)]dt \geq 0 \tag{23}$$

Thus, $\rho_g(Y) > \rho_g(X)$.

If the distortion function $g(t)$ is not strictly concave, that is $g''(t) = 0$ for some t, then the risk measure preserves second order stochastic dominance only weakly, that is we can find Y such that $X \succ_{2nd} Y$ but $\rho(X) = \rho(Y)$.

Theorem 3.2 *A risk measure derived from a distortion function which is concave but not strictly concave does not strongly preserve second order stochastic dominance.*

Proof: The proof parallels the proof of Theorem 3.1; however over any linear portion of the distortion function, we have $g'(S(t)) = M$, a constant, the slope of the linear portion.

For any risk X we can construct a risk Y such that $E[X] = E[Y]$ and $X \succ_{2nd} Y$, and where t_0, the once crossing point, is such that $S_X(t_0) = S_Y(t_0) = b$, and $g(b)$ lies on one linear portion of the distortion function. Also suppose that the linear portion containing $g(b)$ covers the range from $g(a)$ to $g(c)$ where $a < b < c$. Then

$$S_Y(t) \begin{cases} = S_X(t) & \text{for } t \geq t_a \\ = a & \text{for } t = t_a \\ \leq S_X(t) & \text{for } t_a \geq t \geq t_b \\ = b & \text{for } t = t_b \\ \geq S_X(t) & \text{for } t_b \geq t \geq t_c \\ = S_X(t) & \text{for } t \leq t_c \end{cases} \tag{24}$$

which implies that

$$\int_0^{t_c} [S_Y(t) - S_X(t)]dt = 0 \tag{25}$$

and

$$\int_{t_a}^{\infty} [S_Y(t) - S_X(t)]dt = 0. \tag{26}$$

Constructing the same inequalities as in (20), (22), (23), we obtain

$$\rho_g(Z) - \rho_g(X) = g'(S(t_0)) \int_{t_b}^{t_a} [S_Y(t) - S_X(t)]dt \tag{27}$$

and

$$\rho_g(Z) - \rho_g(Y) = g'(S(t_0)) \int_{t_c}^{t_b} [S_X(t) - S_Y(t)]dt \tag{28}$$

which gives,

$$\rho_g(Y) = \rho_g(X). \tag{29}$$

That is, we construct a risk such that $X \succ_{2nd} Y$ but $\rho_g(Y) = \rho_g(X)$.

3.1 Example

A simple example will illustrate the point of this section. Consider two random variables, X and Y:

$$X = \begin{cases} 0 & \text{with probability} \quad 0.95 \\ 50 & \text{with probability} \quad 0.025 \\ 100 & \text{with probability} \quad 0.025. \end{cases}$$

$$Y = \begin{cases} 50 & \text{with probability} \quad 0.975 \\ 100 & \text{with probability} \quad 0.025. \end{cases}$$

Clearly Y has the riskier distribution, and clearly $X \succ_{2nd} Y$.

The CTE with parameter 95% is the expected value of the loss, given the loss lies in the upper 5% of the distribution. In both these cases, the CTE(95%) risk measure is 75. In fact, for any parameter $\alpha \geq 0.95$ the CTE of these two risks will be equal. The CTE risk measure cannot distinguish between these risks in general.

The beta distortion on the other hand will strictly order the risks, for any parameters $0 < a \leq 1$ and $b \geq 1$, provided a and b are not both equal to 1. For example, let $b = 1$ and $a = 0.1$ (this gives the PH-transform risk measure). Then

$$\rho_\beta(X) = 50(.05)^{0.1} + 50(.025)^{0.1} = 71.63$$

$$\text{and} \quad \rho_\beta(Y) = 50 + 50(.025)^{0.1} = 84.58.$$

4 Conclusions

In this paper we have shown that for risk measures which can be expressed in terms of distortion functions, a concave distortion function is a necessary and sufficient condition for coherence and a strictly concave distortion function is a necessary and sufficient condition for strict ordering consistent with second order stochastic dominance.

In this paper, we have shown that the CTE and other partially linear distorted risk measures do not strongly preserve second order stochastic dominance. The beta-distortion risk measure and other risk measures with increasing, strictly concave distortion functions do preserve second order stochastic dominance, and are superior in ordering risk.

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