Expected present value of dividends with a constant barrier in the discrete time model

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Abstract

The process of free reserves in a non-life insurance portfolio as defined in the classical model of risk theory is modified by the introduction of dividend policies that set maximum levels for the accumulation of reserves. The first part of the work formulates the quantification of the dividend payments in the discrete case. The second part presents a solution based on a system of linear equations for discrete dividend payments in the case of a constant dividend barrier, illustrated by solving a specific case.

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1 Introduction

The aim of the present study is to formalize the dividend payment policies in the discrete case for a non-life insurance portfolio, and to obtain the expected present value of the dividend payments.

The classical model analyses the solvency of non-life insurance portfolios using the probability of ruin as the criterion. The discrete case has been studied by various authors, for example, Bowers et al. (1987), Gerber (1988), Michel (1989), Shiu (1989), Willmot (1993,2001), or Li (2002). Section 2 deals with the alternative approach to be found in the literature proposing the pay-out of part of the reserves in the form of dividends (Bühlmann (1970), Gerber (1981), Paulsen (1997), Siegl (1996, 1999)).

Section 3 deals with the analysis of the dividend payments when the model is modified to have a constant dividend barrier \( b(t) = b \), assuming discrete payments, and presents a method for solving such problems. We prove that, as in the continuous case, the probability of ruin is unity. The system of linear equations that allows one to find the expectation of the present value of the dividend payments is obtained, and it is solved using the matrix form of the system. We also include, in section 4, a recursive solution assuming that premium income per unit time is 1 and the distribution of the total cost in a period is concentrated in multiples of the premium.

2 Dividend policy in the discrete case

Following Bühlmann (1970) we take a discrete dividend policy to be that which makes the pay-outs at given times, \( t_i \) for \( i = 1, 2, 3, \ldots \), as long as \( b(t_i) < R(t_i) \), without regard for whether at any intermediate time the level of reserves surpasses the cap represented by the dividend barrier.

Consider the equidistant times \( t_i \) for \( i = 1, 2, 3, \ldots \) with \( t_0 = 0 \), the time unit being one year.

The level of reserves at \( t_i \) before dividend payments, \( R^*(t_i) \), can be defined as

\[
R^*(t_i) = u + c \cdot t_i - S(t_i) - SD(t_{i-1})
\] (1)
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where $S(t_i)$ is the aggregate of claims in the period $[0, t_i]$ , $u$ is the initial reserve at $t_0$, $c$ is the annual premium income, $SD(t_i)$ is the sum of the dividend payments in an interval $[0, t_i]$,

$$SD(t_i) = D_{t_1} + D_{t_2} + \ldots + D_{t_s} \quad \forall t_s \leq t_i$$

$$SD(t_0) = 0$$

and $D_{t_i}$ the dividends paid out at $t_i$ for $i = 1, 2, 3,...$

$$D_{t_i} = \text{Max} \{(R^* (t_i) - b(t_i)), 0\}$$

Let $v$ be a constant annual discount rate for all the periods and $t_k$ the discrete time of ruin. Then

the expected present value of the dividend payments, assuming that there are dividend payments only up to the time of ruin:

$$W = E \left[ \sum_{i=1}^{k} D_{t_i} \cdot v^{t_i} \right] \quad \text{with} \quad t_k = \text{Min} \{t_i / R^* (t_i) < 0\}$$

3 Constant barrier: calculation of $W(u, b)$

We shall now generalize the calculation of the expected present value of the dividends, following the approach of Bühlmann (1970), for the calculation of $W(u, b)$ in a modified model with a constant dividend barrier $b(t) = b$.

We consider $S_i = S(t_i) - S(t_{i-1})$. We assume that $S_i$ are i.i.d. random variables with common probability function $P_s = P[S = s]$ and distribution function $F_S(s) = P[S \leq s]$ for $s = 0, 1, 2, ...$

For simplicity, we redefine $c$ as $c \cdot t_1$ so, for a positive security loading, $E[S] < c$.

The solution of the problem involves considering the random variable of the total accumulated claims in a period as a discrete random variable, and the hypothesis that all monetary magnitudes $(u, b, c, ...)$ are multiples of some given unit. Neither of these conditions implies any major restriction on the validity of the model: in the case of the monetary magnitudes, we simply have to change the reference unit, and in the case of the claims, we shall just have to previously discretize the random variable if it is not already discrete.
In the constant dividend barrier case the probability of ruin is 1 in the continuous case (Eigido dos Reis (1999)). We prove that this is also true in the discrete case. In the discrete case ruin probability is $\psi(u, b) = P[R^* (t_i) < 0]$.

**Theorem 1** Ruin probability in a model with a constant dividend barrier assuming discrete payments is one, $\psi(u, b) = 1$

**Proof.** $\psi(u, b)$ for $u = b$, considering the situation at time $t_1$, is

$$
\psi(b, b) \cdot (1 - F_S(c)) = \sum_{s=c+1}^{b+c} \psi(b + c - s, b) \cdot P_s + 1 - F_S(b + c)
$$

We know that $\psi(b - h, b) \geq \psi(b, b)$, so (2) can be written as

$$
\psi(b, b) \cdot (1 - F_S(c)) \geq \sum_{s=c+1}^{b+c} \psi(b, b) \cdot P_s + 1 - F_S(b + c)
$$

$$
\psi(b, b) \cdot (1 - F_S(b + c)) \geq 1 - F_S(b + c)
$$

and in view of (3), $\psi(b, b) \geq 1$, then

$$\psi(b - h, b) \geq \psi(b, b) \geq 1$$

which implies $\psi(b, b) = 1$. 

Bühlmann (1970) proposed a system of finite difference equations to calculate $W(u, b)$, considering the situation at time $t_1$

$$W(u, b) = \begin{cases} 
0 & \text{si } u < 0 \\
v \cdot \sum_{j=-\infty}^{\infty} W(u + j, b) \cdot P[c - S = j] & \text{si } 0 \leq u \leq b \\
 u - b + W(b, b) & \text{si } u > b
\end{cases}$$

and solving the system for the particular case in which the variation in the reserves is dichotomous, taking only the values $j = -1$ and $j = 1$, with probabilities $p$ and $1-p$. Since the only random factor considered in the model is the occurrence of claims, the case that Bühlmann calculated implies that the claims in a given period can only take the values $(c + 1)$ and $(c - 1)$, with probabilities $p$ and $1-p$. 

3
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To generalize the calculation of $W(u, b)$, we shall analyse the situation of the process at time $t_1$:

The dividend payments will depend on whether $R^* (t_1)$ is greater or less than the level of the barrier $b$. Hence, $R^* (t_1) = u + c - s$

- **Case 1:** $R^* (t_1) = u + c - s$ is greater than the level of the barrier $b$. Graphically,

  \[ D_{t_1} = u + c - s - b \]

  In this case, the dividend payments in $t_1$, $D_{t_1} = SD(t_1)$, are positive, with their amount being the difference between $R^* (t_1)$ and the barrier $b$, i.e. $D_{t_1} = u + c - s - b$. Also, for the calculation of $W(u, b)$ the calculated future dividends must be discounted to $t_1$, and are $W(b, b)$

- **Case 2:** $R^* (t_1)$ is less than or equal to the level of the barrier $b$, independently of what happened in the interval $[0, t_1]$. Graphically, In this case, for the calculation of $W(u, b)$, we must discount $W(u + c - s, b)$
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To determine the expression for the expected present value of the dividend payments, we shall formalize the two cases described previously section, by setting up a system of linear equations.

According to the initial level of reserves $u$, such that $u \leq b$, one can define $b + 1$ equations for the calculation of $W(u, b)$ with $u = 0, \ldots, b$.

**Theorem 2** For $x = 0, 1, \ldots, c, c + 1, \ldots, b$

$$W(b - x, b) = v \cdot \left[ W(b, b) \cdot F_S(c - x) + \sum_{s=0}^{c-(x+1)} (c - s - x) \cdot P_s + \sum_{s=1}^{b} W(b - s, b) \cdot P_{s+c-x} \right] \quad (4)$$

**Proof.**

- **If the initial level of reserves coincides with the barrier level, $u = b$**

First, let us consider the case in which the total of claims $s$ coincides with the premium income $c$. At $t_1$ therefore, the new level of reserves is $b + c - s = b$, hence

$$W(b, b) \cdot P_c \quad (5)$$

In those cases when the amount of claims $s$ lies in the interval $[0, c - 1]$, there will be dividend payments, since $b + c - s$ is greater than $b$, with $D_{t_1} = c - s$, so that the new level of reserves will be $u = b$. Hence

$$\sum_{s=0}^{c-1} W(b + c - s, b) \cdot P_s = \sum_{s=0}^{c-1} (W(b, b) + (c - s)) \cdot P_s$$
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or, equivalently,

\[
\sum_{s=0}^{c-1} (W(b,b) + (c-s)) \cdot P_s = W(b,b) \cdot \sum_{s=0}^{c-1} P_s + \sum_{s=0}^{c-1} (c-s) \cdot P_s = W(b,b) \cdot F_S(c-1) + \sum_{s=0}^{c-1} (c-s) \cdot P_s \tag{6}
\]

Finally, let us consider the cases in which the aggregate claims amount \(s\) lies in the interval \([c+1, b+c]\). The level of reserves at \(t_1\), \(b+c-s\), is less than \(b\). Hence

\[
\sum_{s=c+1}^{b+c} W(b+c-s,b) \cdot P_s
\]

Applying the change of variable \(r = s - c\), one has

\[
\sum_{s=c+1}^{b+c} W(b+c-s,b) \cdot P_s = \sum_{r=1}^{b} W(b-r,b) \cdot P_{r+c} \tag{7}
\]

We can therefore write \(W(b,b)\) as the discounted sum of \((5), (6)\) and \((7)\)

\[
W(b,b) = v \cdot \left[ W(b,b) \cdot F_S(c) + \sum_{s=0}^{c-1} (c-s) \cdot P_s + \sum_{s=1}^{b} W(b-s,b) \cdot P_{s+c} \right] \tag{8}
\]

- **If the initial level of reserves is below the barrier by less than \(c\) units, \(b-c < u < b\).**

The equation for \(u = b-x\), when \(x = 1, ..., c-1\) results from taking into account that the new level of reserves at \(t_1\) is

\[
b - x + c - s \tag{9}
\]

If \((9)\) is greater than \(b\) then \(s < c - x\), and therefore leads to dividend payment. This is reflected in the sum \(\sum_{s=0}^{c-(x+1)} W(b-x+c-s,b) \cdot P_s\), where \((c-s-x)\) would have to be paid out, leaving the new level of reserves at \(b\). Hence

\[
\sum_{s=0}^{c-(x+1)} ((c-s-x) + W(b,b)) \cdot P_s = W(b,b) \cdot F_S(c-(x+1)) + \sum_{s=0}^{c-(x+1)} (c-s-x) \cdot P_s \tag{10}
\]

If \((9)\) is less than \(b\), then \(s > c - x\). In this case, there will be no dividend payment. Also, so as not to cause ruin, one must have that \(b-x+c-s \geq 0 \Rightarrow s \leq b-x+c\). Hence, the amount of \(s\) has to lie in the interval \([c-x+1, c-x+b]\).

\[
\sum_{s=c-x+1}^{c-x+b} W(b-x+c-s,b) \cdot P_s \tag{11}
\]
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With the change of variable $r = s - (c - x)$ \(\Rightarrow s = r + (c - x)\), expression (11) becomes

$$\sum_{r=1}^{b} W(b-r,b) \cdot P_{r+c-x}$$

(12)

Lastly, it can be assumed that $b - x + c - s = b \Rightarrow s = c - x$, in which case one has

$$W(b,b) \cdot P_{c-x}$$

(13)

Grouping together (10), (12) and (13), one then has

$$W(b-x,b) = v \cdot \left[ W(b,b) \cdot F_s(c-(x+1)) + \sum_{s=0}^{c-(x+1)} (c-s-x) \cdot P_{s+} + \sum_{r=1}^{b} W(b-r,b) \cdot P_{r+c-x} \right]$$

$$+ W(b,b) \cdot F_s(c-x) + \sum_{s=0}^{c-(x+1)} (c-s-x) \cdot P_{s+} + \sum_{s=1}^{b} W(b-s,b) \cdot P_{s+c-x}$$

(14)

- The initial level of reserves is below the barrier by at least $c$ units, $0 \leq u \leq b - c < b$.

Now, for $u = b - x$, when $x = c, c+1, \ldots, b$, the new level of reserves is $b - x + c - s$, which is therefore always less than $b$ given the values of $x$. There is therefore no dividend payment:

$$W(b-x,b) = v \cdot \sum_{s=0}^{b+(c-x)} W(b-x+c-s,b) \cdot P_s$$

(15)

where the top of the sum limits the case in which the new level of reserves is negative, $b - x + c - s \geq 0 \Rightarrow s \leq b + (c - x)$

We can observe that the expressions (8), (14) and (15) are included in the general expression (4).

3.1 Matrix form of the system

It can be readily verified that the generalization of the system presented in the previous subsection, and defined by equations (8), (14) and (15), can be written in matrix form:

$$v \cdot A \cdot \bar{w} + v \cdot D = \bar{w}$$

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where $A$ is the matrix of coefficients made up of different submatrices:

$$A = \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix}$$

with:

- $M_1$ is a vector of $(c + 1)$ components. We know that $E[S] < c$, then $F_S(c - 1) > 0$
- $M_2$ is a matrix of order $(c + 1) \times b$
- $M_3$ is a null vector of $(b - c)$ components
- $M_4$ is a matrix of order $(b - c) \times b$

The matrix $A$ is therefore a square matrix of order $(b + 1)$,

$$A = \begin{pmatrix} F_s(c) & P_{c+1} & P_{c+2} & \cdots & \cdots & P_{c+b} \\ F_s(c-1) & P_c & P_{c+1} & P_{c+2} & \cdots & \cdots & P_{c+b-1} \\ F_s(c-2) & P_{c-1} & P_c & P_{c+1} & \cdots & \cdots & P_{c+b-2} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \cdots & \cdots \\ F_s(0) & P_1 & P_2 & P_3 & \cdots & \cdots & P_b \\ 0 & P_0 & P_1 & P_2 & \cdots & \cdots & P_{b-1} \\ 0 & 0 & P_0 & P_1 & \cdots & \cdots & P_{c+b-1} \\ 0 & 0 & 0 & P_0 & \cdots & \cdots & P_{c+b-2} \\ 0 & 0 & 0 & 0 & \cdots & \cdots & P_{c+b-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & P_0 & P_c \end{pmatrix}$$

The vector of independent terms $D$ is of order $(b + 1) \times 1$, formed by $c$ first elements different from zero, and the remaining $b + 1 - c$ elements equal to zero. $\overline{w}$ is the vector of $b + 1$ unknowns,
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\[
D = \begin{pmatrix}
\sum_{s=0}^{c-1} (c-s) \cdot P_s \\
\sum_{s=0}^{c-2} (c-s-1) \cdot P_s \\
\sum_{s=0}^{c-3} (c-s-2) \cdot P_s \\
\vdots \\
P_0 \\
0 \\
\vdots \\
0
\end{pmatrix}
\]

\[
\varpi = \begin{pmatrix}
W(b,b) \\
W(b-1,b) \\
W(b-2,b) \\
\vdots \\
W(b-c,b) \\
W(b-c-1,b) \\
\vdots \\
W(0,b)
\end{pmatrix}
\]

The solution of system (16) is,

\[
\varpi = [I - v \cdot A]^{-1} \cdot v \cdot D
\]

for \([I - v \cdot A]\) regular.

The specific case indicated in Section 3, where the aggregate of claims in a period can only take the values \((c+1)\) and \((c-1)\), with probabilities \(p\) and \(q = 1-p\), can be solved by means of a system of finite difference equations (Bühlmann (1970)), and alternatively by applying the matrix system presented in the previous subsection. For a positive security loading \(p < q\).

Thus, for instance, taking \(b = 5\) and \(c = 1\), the matrix system is:

\[
\begin{pmatrix}
W(5,5) \\
W(4,5) \\
W(3,5) \\
W(2,5) \\
W(1,5) \\
W(0,5)
\end{pmatrix}
= \begin{pmatrix}
P_0 + P_1 & P_2 & P_3 & P_4 & P_5 & P_6 \\
P_0 & P_1 & P_2 & P_3 & P_4 & 0 \\
0 & P_0 & P_1 & P_2 & P_3 & 0 \\
0 & 0 & P_0 & P_1 & P_2 & 0 \\
0 & 0 & 0 & P_0 & P_1 & P_2 \\
0 & 0 & 0 & 0 & P_0 & P_1
\end{pmatrix}
\begin{pmatrix}
P_0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{pmatrix}
\]

where \(P_0 = 1-p\), \(P_2 = p\) and \(P_i = 0 \forall i \neq 0,2\).
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4 Analysis of the $c = 1$ case

If we assume the $c = 1$ case, it is possible to calculate $W(u, b)$ as a recursive process.

As $E[S] < c = 1$, then $P_0 > 0$.

**Theorem 3** For $x = 0, 1, ..., b - 1$

$$W(b - x, b) = \frac{1}{C_1(x)} \left( C_3(x) + \sum_{s=x+1}^{b} W(b - s, b) \cdot C_2(s, x) \right)$$

where $C_1(x), C_2(s, x)$ and $C_3(x)$ are calculated in a recursive form,

$$C_1(x + 1) = C_1(x) \cdot (1 - v \cdot P_1) - v \cdot P_0 \cdot C_2(x + 1, x)$$

$$C_2(s, x + 1) = C_2(s, x) \cdot v \cdot P_0 + C_1(x) \cdot v \cdot P_{s-x}, s = x + 2, ..., b$$

$$C_3(x + 1) = v \cdot P_0 \cdot C_3(x) = v^{x+1} \cdot P_0^{x+1}$$

where

$$C_1(0) = 1 - v \cdot P_1$$

$$C_2(s, 0) = v \cdot P_{s+1}, s = 1, ..., b$$

$$C_3(0) = v \cdot P_0$$

and

$$W(0, b) = \frac{C_3(b)}{C_1(b)}$$

**Proof.** For $u = b$, we can obtain

$$W(b, b) = v \left[ P_0 + W(b, b) \cdot P_0 + W(b, b) \cdot P_1 + \sum_{s=2}^{b+1} W(b + 1 - s, b) \cdot P_s \right]$$

or, equivalently,

$$W(b, b) = W(b - x, b)$$

$$W(0, b) = \frac{1}{1 - v \cdot P_1 - v \cdot P_0} \cdot \left( v \cdot P_0 + v \cdot \sum_{s=1}^{b} W(b - s, b) \cdot P_{s+1} \right)$$

(17)

$W(b - x, b)$ take the form,

$$W(b - x, b) = v \left[ W(b - x + 1, b) \cdot P_0 + W(b - x, b) \cdot P_1 + \sum_{s=2}^{b-x+1} W(b - x + 1 - s, b) \cdot P_s \right]$$

(18)

that for $x = 1$

$$W(b - 1, b) = \frac{v}{1 - v \cdot P_1} \left( W(b, b) \cdot P_0 + \sum_{s=2}^{b} W(b - s, b) \cdot P_s \right)$$

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If we substitute (17) in (19), and simplifying terms gives

$$
W(b - 1, b) = \frac{v^2 \cdot P_0^2 + \sum_{s=2}^{b} W(b - s, b) \cdot (v^2 \cdot P_0 \cdot P_{s+1} + v \cdot P_s \cdot (1 - v \cdot P_1 - v \cdot P_0))}{(1 - v \cdot P_1) \cdot (1 - v \cdot P_1 - v \cdot P_0) - v^2 \cdot P_0 \cdot P_2}
$$

(20)

Putting

$$
C_1(1) = (1 - v \cdot P_1) \cdot (1 - v \cdot P_1 - v \cdot P_0) - v^2 \cdot P_0 \cdot P_2
$$

$$
C_2(s, 1) = v^2 \cdot P_0 \cdot P_{s+1} + v \cdot P_s \cdot (1 - v \cdot P_1 - v \cdot P_0), \forall s > 1
$$

$$
C_3(1) = v^2 \cdot P_0^2
$$

(20) can be re-written as

$$
W(b - 1, b) = \frac{1}{C_1(1)} \cdot \left( C_3(1) + \sum_{s=2}^{b} W(b - s, b) \cdot C_2(s, 1) \right)
$$

and generalized to,

$$
W(b - x, b) = \frac{1}{C_1(x)} \cdot \left( C_3(x) + \sum_{s=x+1}^{b} W(b - s, b) \cdot C_2(s, x) \right)
$$

(21)

Now we show (21) by induction. We assume (21) is true for $x$, and we show that is true for $x+1$. From

(18) for $x+1$,

$$
(1 - v \cdot P_1) \cdot W(b - x - 1, b) = v \cdot \left[ W(b - x, b) \cdot P_0 + \sum_{s=1}^{b-x-1} W(b - x - 1 - s, b) \cdot P_{s+1} \right]
$$

(22)

We substitute (21) in (22),

$$
W(b - x - 1, b) = \frac{v \cdot P_0 \cdot C_3(x) + \sum_{s=x+2}^{b} W(b - s, b) \cdot (v \cdot P_0 \cdot C_2(s, x) + C_1(x) \cdot v \cdot P_{s-x})}{C_1(x) \cdot (1 - v \cdot P_1) - v \cdot P_0 \cdot C_2(x + 1, x)}
$$

(23)

which implies

$$
W(b - (x + 1), b) = \frac{1}{C_1(x + 1)} \cdot \left( C_3(x + 1) + \sum_{s=x+2}^{b} W(b - s, b) \cdot C_2(s, x + 1) \right)
$$

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so (21) is demonstrated.

From (23) and (24), we obtain the recursive formula for \( C_1(x), C_2(s,x) \) and \( C_3(x) \)

\[
C_1(x + 1) = C_1(x) \cdot (1 - v \cdot P_1) - v \cdot P_0 \cdot C_2(x + 1, x)
\]

\[
C_2(s, x + 1) = v \cdot P_0 \cdot C_2(s, x) + C_1(x) \cdot v \cdot P_{s-x}, \quad s = x + 2, ..., b
\]

\[
C_3(x + 1) = v \cdot P_0 \cdot C_3(x)
\]

where the initial values are obtained from (17) and (21),

\[
C_1(0) = 1 - v \cdot P_1 - v \cdot P_0
\]

\[
C_2(s, 0) = v \cdot P_{s+1}
\]

\[
C_3(0) = v \cdot P_0
\]

To calculate \( W(0, b) \), we write (18), for \( x = b \) :

\[
(1 - v \cdot P_1) \cdot W(0, b) = v \cdot W(1, b) \cdot P_0
\]

(25)

and (21) for \( x = b - 1 \) :

\[
W(1, b) \cdot C_1(b - 1) = C_3(b - 1) + W(0, b) \cdot C_2(b, b - 1)
\]

(26)

From (25) and (26) it is obtained

\[
W(0, b) = \frac{v \cdot P_0 \cdot C_3(b - 1)}{C_1(b - 1) \cdot (1 - v \cdot P_1) - v \cdot P_0 \cdot C_2(b, b - 1)} = \frac{C_3(b)}{C_1(b)}
\]

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