Estimates for the Ruin Probability in the Classical Risk Model with Constant Interest Force in the Presence of Heavy Tails

D.G. Konstantinides\(^a\), Q.H. Tang\(^b\)\(^\gamma\), G.Sh. Tsitsiashvili\(^c\)\(^z\)

\(^a\) Department of Statistics and Actuarial Science, University of the Aegean, Samos, 83200 Greece
\(^b\) Department of Quantitative Economics, University of Amsterdam, Roetersstraat 11, 1018 WB Amsterdam, The Netherlands
\(^c\) Institute of Applied Mathematics FEB RAS, Radio str. 7, Vladivostok, 690041 Russia

Abstract

In this paper we investigate the ruin probability in the classical risk model under a positive constant interest force. We restrict ourselves to the case where the claim size is heavy-tailed, i.e. the equilibrium distribution function of the claim size belongs to a wide subclass of the subexponential distributions. Two-sided estimates for the ruin probability are developed by reduction from the classical model without interest force.

Key words and phrases: Classical risk model; Constant interest force; The Ruin probability; Subexponential distribution.

1 Introduction

The classical risk model with homogeneous Poisson arrival process, constant premium rate and constant interest force has been investigated by many authors such as Sundt & Teugels (1995, 1997), Asmussen (1998), Klüppelberg & Stadtmüller (1998) and Kalashnikov & Konstantinides (2000). We address in the present paper two-sided bounds for the ruin probability in this model. The well-known inequalities in the classical risk model without interest force enable us to derive accurate two-sided estimates. The idea of the reduction is not new, but only recently this method became effective as the necessary tools were accumulated. This approach is also applicable to the study of the convergence rate of the ruin probability approximations.

\(^\gamma\)Corresponding author.

\(^z\)E-mail addresses: konstant@aegean.gr (D.G. Konstantinides); qtang@fee.uva.nl or tangqihe@263.net (Q.H. Tang); guram@iam-mail.febras.ru (G.Sh. Tsitsiashvili).
We suppose that the claims sizes, \((Z_k)_{k \geq 1}\), form a sequence of i.i.d. non-negative r.v.’s, with a common d.f. \(B(x) = \Pr(Z_1 \leq x), x \geq 0\), and a finite expectation \(b\). Throughout this paper, the d.f. \(B\) always satisfies \(B(x) > 0\) for all \(x \geq 0\). We denote by
\[
F(x) = \frac{1}{b} \int_0^x B(z) \, dz; \quad x \geq 0;
\]
the equilibrium distribution function (e.d.f.) of the d.f. \(B\). We assume that, as usual, the claim arrival times constitute a homogeneous Poisson process \((N(t))_{t \geq 0}\), which is independent of \((Z_k)_{k \geq 1}\) and has an intensity \(\lambda > 0\). Therefore, the compound Poisson process \(X(t) = \sum_{k=1}^{N(t)} Z_k\) represents the total claim amount accumulated up to time \(t \geq 0\), with \(X(t) = 0\) when \(N(t) = 0\). We write \(\gamma = b\) and let \(c\) be the constant gross premium rate which is not necessarily positive. We assume that there exists a constant interest force \(r > 0\) which affects the risk process. Let \(u > 0\) be the initial surplus of the insurance company, then the total surplus up to time \(t\), represented by \(U_r(t)\), satisfies the equation
\[
U_r(t) = u e^{rt} + c \int_0^t e^{r(t-z)} \, dX(z); \quad t \geq 0;
\]
The ultimate ruin probability for this risk process is then defined by
\[
\overline{\Phi}(u) = \Pr \left( \inf_{t \geq 0} U_r(t) < 0 \mid U_r(0) = u \right); \quad u > 0;
\]
As many recent references in the fields of risk theory, we are interested in heavy-tailed claim sizes. The most important class of heavy-tailed d.f.’s is the subexponential class \(S\). By definition, a d.f. \(F\) supported on \([0, 1)\) belongs to the class \(S\) if for any (or equivalently for some) \(n \geq 2\), it holds that
\[
\lim_{x \to \infty} \frac{F^{*n}(x)}{F(x)} = n;
\]
where \(F^{*n}\) denotes the \(n\)-fold convolution of the d.f. \(F\). We refer to Embrechts et al. (1997), Rolski et al. (1999) and Asmussen (2000) for thorough reviews of the applications of the class \(S\) to insurance and finance.

It is well-known that, if the e.d.f. \(F\) of the claimsize belongs to the class \(S\) and the safety loading condition \(\gamma \leq c\) holds, then
\[
\overline{\Phi}_0(u) = \frac{\gamma}{c} \overline{F}(u); \quad u > 1;
\]
See Feller (1971) and Embrechts & Veraverbeke (1982). Based on the classical asymptotics (1.2), Kalashnikov & Tsitsiashvili (1999, 2000) and Mikosch & Nagaev (2001) introduced as an auxiliary function the relative error of the approximation (1.2)
\[
\xi(u) = \frac{\overline{\Phi}_0(u)}{\overline{F}(u)}; \quad u > 1;
\]
Then they studied the bounds and the convergence rate of the quantity $\zeta(u) \equiv 0$ as $u \to 1$.

Recently, Asmussen (1998), Klüppelberg & Stadtmüller (1998) and Kalashnikov & Konstantinides (2000) considered the ruin probability $\bar{\Lambda}(u)$ with the constant interest force $r > 0$. They established an asymptotic formula which is similar as the classical asymptotics (1.2).

In this paper we continue the work of the cited papers. Asmussen’s formula for the ruin probability $\bar{\Lambda}(u)$ is the starting point of our present investigation. First, in Section 2, after some analysis on Asmussen’s formula, we introduce an additional condition on the claims size distribution. By this condition we characterize a new broad subexponential subclass. Then, based on Asmussen’s formula we drive some two-sided bounds for the ruin probability $\bar{\Lambda}(u)$ in Section 3. Some examples are laid in Section 4, and numerical results are in Section 5.

2 Main Results

2.1 On an Asymptotic Formula for the Ruin Probability

Recall that $F$ represents the e.d.f. of the claims size distribution $B$. Under the assumption $F \in S$, Asmussen (1998) first established an asymptotic formula for the ruin probability $\bar{\Lambda}(u)$ with $r > 0$ that

$$\bar{\Lambda}(u) \approx \frac{Z_1}{r} \frac{\bar{B}\left(\frac{z}{u}\right)}{z}; \quad u \to 1.$$  (2.1)

Almost simultaneously, Klüppelberg & Stadtmüller (1998) used sophisticated analytical arguments to derive (2.1) in the presence of claims size having regularly varying tail with exponent $\beta > 1$; i.e.

$$\bar{B}(x) = x^{\beta} L(x); \quad x > 0;$$

where $L(x)$ is a positive function which is slowly varying as $x$ tends to infinity (we rewrite $B \in R_\beta$ as usual). Later, Kalashnikov & Konstantinides (2000) provided a simple proof for Asmussen’s formula (2.1) for the whole subexponential case. In the proofs provided by Asmussen (1998) and Kalashnikov and Konstantinides (2000), however, some supporting arguments should be required. This was pointed out by Asmussen et al. (2002) in pages 403 and 404. We also refer readers to the proof of Lemma 3 in Kalashnikov and Konstantinides (2000), where the authors used an implication that, for in:nitesimal quantities $A_i(u)$, 1 \cdot i \cdot 4, if $A_1(u) \approx A_2(u)$ and $A_3(u) \approx A_4(u)$ then

$$A_1(u) \approx A_3(u) \approx A_2(u) \approx A_4(u);$$

which is wrong, in general.

We prove that (2.1) remains valid under an additional restriction. That is:
Theorem 2.1. In the classical risk model with a constant interest force \( r > 0 \), the asymptotics (2.1) is true if the e.d.f. \( F \in S \) and that, for some \( v > 1 \);

\[
\limsup_{x \to 1} \frac{F(vx)}{F(x)} < 1;
\]  
(2.2)

Clearly, if the claimsize distribution \( B \in R \) for some \( \gamma > 1 \) then the e.d.f. \( F \) satisfies all the conditions asked in Theorem 2.1.

Motivated by Theorem 2.1 we introduce a new subclass of subexponential d.f.’s below:

Definition. Let \( F \) be a d.f. supported on \([0; 1)\). We say that \( F \) belongs to \( A \) if \( F \in S \) and (2.2) holds for some \( v > 1 \).

We point out that the class \( A \) covers almost all the well-known subexponential d.f.’s. In fact, by its definition one easily checks that

Remark. All the d.f.’s listed in Table 1.2.6 in Embrechts et al. (1997) belong to the class \( A \). Namely the Pareto, the Lognormal, the Weibull, the Loggamma, the Burr, the Benktander I and II distributions are included in the class \( A \).

With the notation

\[
D(x) = 1 \sum_{x}^{\infty} x F(z) \frac{dz}{z^2}, \quad x > 0;
\]

we immediately obtain from Theorem 2.1 that

Corollary. In the classical risk model with a constant interest force \( r > 0 \), the asymptotic formula

\[
\overline{\mathcal{A}}_r(u) = \frac{\nu}{ru} D(u) \overline{F}(u); \quad u > 1;
\]  
(2.3)

is true if the e.d.f. \( F \in A \).

Proof. From (2.1) we have

\[
\overline{\mathcal{A}}_r(u) = \frac{1}{r} \sum_{u}^{\infty} \frac{B(z)}{z^2} \frac{dz}{z} = \frac{1}{r} \sum_{u}^{\infty} \frac{1}{z} \overline{F}(z);
\]

the right-hand side of which equals to \( \frac{\nu}{ru} D(u) \overline{F}(u) \) by integration by parts.

2.2 Two-sided Estimates for the Ruin Probability

In order to investigate two-sided estimates for the ruin probability \( \overline{\mathcal{A}}_r(u) \) with \( r > 0 \), now we seek for another auxiliary function which plays a similar role in the present situation as that
of $\zeta(u)$ in the case without interest force. The asymptotic formula (2.3) urges us to use

$$i(u) = \frac{\tilde{A}_r(u)}{\frac{ru}{\mu} D(u) F(u)} \cdot \zeta(u) \cdot \tilde{A}_0(u) \cdot \frac{1}{\mu} \cdot \frac{1}{\tilde{A}_0(u)} \cdot \frac{1}{\mu} \cdot \frac{1}{\tilde{A}_0(u)}$$

Obviously, $i(u)$ represents the relative error of the approximation (2.3).

Now we state the main result of the paper:

Theorem 2.2. In the classical risk model, if $\frac{1}{2} < c$, then for any $u > 0$ we have that

$$i_i(u) \cdot i(u) \cdot i_+(u);$$

(2.4)

where,

$$i_i(u) = i \left[ \frac{1}{D(u)} \cdot \frac{\mu}{c + ru} \cdot \frac{\zeta(u) + \tilde{A}_0(u)}{\tilde{A}_0(u)} \right];$$

$$i_+(u) = i \left[ \frac{1}{D(u)} \cdot \frac{\mu}{c + ru} \cdot \frac{\zeta(u) + \tilde{A}_0(u)}{\tilde{A}_0(u)} \right];$$

and $\zeta(u) = \sup_{x, u} \zeta(x)$.

The following bound for $|i(u)|$ is sometimes more convenient for applications:

Corollary. Under the conditions of Theorem 2.2, we have that, for any $u > 0$,

$$j_i(u) \cdot \frac{1}{D(u)} \cdot \frac{\mu}{c + ru} \cdot \frac{\zeta(u) + \tilde{A}_0(u)}{\tilde{A}_0(u)} \cdot \frac{1}{\mu} \cdot \frac{1}{\tilde{A}_0(u)}.$$ 

Proof. The proof is straightforward from (2.4).

Note that in Theorem 2.2 and its corollary, we did not require any condition on the tail behavior of the claims size. Now we assume that (2.2) holds for some $\nu > 1$. By Lemma 3.3 below we know that (2.2) holds for any $\nu > 1$. Thus, for any $\nu > 1$, there exists some $l(\nu) = l_F(\nu) > 0$ such that

$$d(\nu) = \sup_{x > l(\nu)} \frac{F(vx)}{F(x)} < 1.$$ 

For convenience we write $d(1) = 1$. With the $\nu, l(\nu)$ and $d(\nu)$ given above, we introduce

$$d(\nu; k) = \sup_{x > l(\nu)} \frac{F(vx)}{F(x)}; \quad \gamma(\nu) = \frac{1}{k} \cdot d(\nu; k).$$ 

Obviously, it holds for any $\nu > 1$ that $d(\nu; 0) = d(1) = 1$ and $d(\nu; 1) = d(\nu)$. We obtain:
Theorem 2.3. In addition to the conditions of Theorem 2.2, we assume that (2.2) holds for some \( v > 1 \). Then the following inequalities hold for any \( v > 1 \) and \( u > l(v) \):

\[
\begin{align*}
 i_i(u) & : - \frac{1}{3v} \frac{c}{c + ru} + \frac{\zeta(u)}{A} + \frac{\mu}{c_1} \frac{F(u)}{1 + \zeta(u)} [1 + \zeta(u)] \geq 0; \\
 i + (u) & : - \frac{1}{3v} \frac{c}{c + ru} + \frac{\zeta(u)}{A} + \frac{\mu}{c_1} \frac{F(u)}{1 + \zeta(u)} [1 + \zeta(u)] \geq 0; \\
 j_i(u) & : - \frac{1}{3v} \frac{c}{c + ru} + \frac{\zeta(u)}{A} + \frac{\mu}{c_1} \frac{F(u)}{1 + \zeta(u)} [1 + \zeta(u)] \geq 0.
\end{align*}
\]

We notice that all the bounds given in Theorem 2.3 essentially depend on three quantities \( c + ru \), \( \zeta(u) \) and \( F(u) \). This enables us to investigate the convergence rate of the deviation \( i_i(u) \) to 0 as \( u \rightarrow 0 \) by reduction from the existing results in the literature. We refer readers to Kalashnikov & T. Sitsiashvili (2000) for some estimates for \( \zeta(u) \) or \( A_0(u) \), and to Mikosch & Nagaev (2001) for some details on the convergence rate of \( \zeta(u) \) to 0 as \( u \rightarrow 0 \).

3 Proofs of the Main Results

3.1 Some Lemmas

In this subsection we propose some lemmas about the class \( A \). They will play crucial roles in the proofs of our main results.

Lemma 3.1. Let \( F \) be a d.f. supported on \([0; 1]\). If (2.2) holds for some \( v > 1 \), then, for all \( x \leq l(v) \):

\[
D(x) - \frac{1}{3v} - \frac{1}{c + ru} \geq 0; \quad \frac{d(v)}{v} > 0; \quad 1 \leq v > 1.
\]

where the notations involved were given in Section 2.

Proof. Recall the definitions of \( d(v) \); \( d(v; k) \) and \( \frac{1}{3v} \). We have that the quantity \( d(v; k) \) is non-decreasing in \( k \geq 0 \). Therefore,

\[
\frac{1}{3v} \frac{d(v; 0)}{v} \frac{d(v; 1)}{v} = \frac{1}{i} \frac{d(v)}{v} > 0; \quad v > 1.
\]

As for the first inequality in (3.1), we have

\[
D(x) - \frac{1}{3v} - \frac{x}{F(x)} \sum_{k=1}^{\mu} \frac{1}{v^k} Z_{v^k} \frac{dz}{v^k} - \frac{x}{v^k} \sum_{k=1}^{\mu} \frac{1}{v^k} \frac{1}{v^k} = \frac{1}{3v}:
\]

This ends the proof. \( \Box \)
Lemma 3.2. Let \( F \) be a d.f. supported on \([0; 1)\). Then
\[
A_F = \limsup_{x \to 1} \frac{1}{x} \int_1^x \frac{F(z)}{z} \, dz < 1
\]  
(3.3)
if and only if (2.2) holds for some \( v > 1 \).

Proof. The proof of the “if” assertion follows from Lemma 3.1. Now we only need to prove the “only if” assertion. Clearly, for any \( v > 1 \),
\[
\frac{x}{F(x)} \int_1^x \frac{F(z)}{z} \, dz, \quad \frac{x}{F(x)} \int_1^x \frac{F(z)}{z} \, dz, \quad \frac{x}{F(x)} \int_1^x \frac{F(vx)}{z} \, dz < \frac{1}{v} \frac{1}{vx}.
\]
It follows that
\[
\limsup_{x \to 1} \frac{F(vx)}{F(x)} \cdot \frac{v}{v} \limsup_{x \to 1} \frac{x}{F(x)} \int_1^x \frac{F(z)}{z} \, dz = \frac{v}{v} A_F; \quad v > 1.
\]
So (2.2) holds for all \( v > 1 \). This ends the proof. \( \qed \)

Lemma 3.3. Let \( F \) be a d.f. supported on \([0; 1)\) with a density function \( f(x) \) which is eventually non-increasing. Then the following statements are equivalent:

I. (2.2) holds for some \( v > 1 \);

II. (2.2) holds for any \( v > 1 \);

III. the hazard rate of \( F \), \( q(x) = f(x)/F(x) \), satisfies
\[
\liminf_{x \to 1} xq(x) > 0;
\]  
(3.4)

Proof. We prove the lemma by the following order of implications: I \( \Rightarrow \) III \( \Rightarrow \) II \( \Rightarrow \) I:

(1). I \( \Rightarrow \) III: For the fixed \( v > 1 \) in I and all sufficiently large \( x > 0 \),
\[
\frac{F(vx)}{F(x)} = 1 + x F'(t) \, dt \, F(x), \quad 1 \int_1^x (v), 1xq(x);
\]
from which it follows that
\[
\liminf_{x \to 1} xq(x), \quad \frac{1}{v} \int_1^x \liminf_{x \to 1} 1 i \frac{F(vx)}{F(x)} > 0;
\]

(2). III \( \Rightarrow \) II: For any fixed \( v > 1 \) and all large \( x > 0 \),
\[
\frac{F(vx)}{F(x)} = R_{vx} f(t) \, dt + F(vx) / (v 1) x f(vx) + F(vx) = 1 / (v 1) xq(vx) + 1;
\]
which, together with (3.4), implies that (2.2) holds for any \( v > 1 \).

(3). II \( \Rightarrow \) I: This step is trivial. \( \qed \)
Remark. Let $F$ be the e.d.f. of the claimsize distribution $B$. We write

$$L_1(v) = \liminf_{x \to 1} \frac{\mathbb{F}(vx)}{\mathbb{F}(x)} \quad \text{and} \quad L_2(v) = \limsup_{x \to 1} \frac{\mathbb{F}(vx)}{\mathbb{F}(x)}; \quad v > 1,$$

Clearly, Lemma 3.3 indicates that if $L_2(v) = 1$ for some $v > 1$ then it holds for all $v > 1$. Furthermore, going along the line of the proof of Lemma 3.3 we also obtain that

$$L_1(v) < 1 \quad 9v > 1 \quad (\quad L_1(v) < 1 \quad 8v > 1 \quad (\quad \limsup_{x \to 1} xq(x) > 0:\n
So we have that if $L_1(v) = 1$ for some $v > 1$ then it holds for all $v > 1$. From these discussions we can classify all possibilities of the values of $L_1(v)$ and $L_2(v)$ into three cases:

1. $L_1(v) = L_2(v) = 1 \quad 8v > 1$;
2. $L_1(v) < 1$ but $L_2(v) = 1 \quad 8v > 1$;
3. $L_2(v) < 1 \quad 8v > 1$.

The first case indicates that $F(x)$ is slowly varying as $x \to 1$. The third case is just the fundamental assumption of the present paper. Our past experience shows that problems for the second case are often very complicated.

### 3.2 Proof of Theorem 2.1

We shall need two auxiliary functions

$$G_r(u) = 1 - \frac{\bar{A}_r(u)}{A_r(0)}; \quad k_r(u) = \frac{Z_1}{u} z dG_r(z); \quad u \geq 0;$$

and a notation

$$K_r = \frac{1}{2} \frac{A_r(0)}{A_r(0)};$$

which were first introduced by Sundt & Teugels (1995). These expressions enable us to take the following representation for the ruin probability:

$$\bar{A}_r(u) = \frac{\mu}{K_r + \frac{1}{2}} \frac{k_r(u)}{u} \frac{Z_1}{u} k_r(z) \frac{dz}{z^2}; \quad u \geq 0; \quad (3.5)$$

Further we shall use the following two-sided bounds of $k_r(u)$:

$$\frac{1}{2} r + K_r \frac{\mu}{c + ru} \mathbb{F}(u) \cdot k_r(u) \cdot \frac{1}{2} K_r \frac{(c + ru)}{r^{1/2}} \frac{\bar{A}_r(u)}{A_0(u)}; \quad (3.6)$$

See Kalashnikov & Konstantinides (2000).

We assume temporarily that the safety loading condition $\frac{1}{2} < c$ holds. Therefore the classical formula (1.2) is valid. The inequalities (3.6), together with the asymptotics (1.2), give the relationship

$$k_r(u) \gg \frac{1}{2} r + K_r \mathbb{F}(u); \quad u \to 1; \quad (3.7)$$
Hence,
\[ \limsup_{u \to 1} \frac{R_1 k_r(z) dz}{u} = \limsup_{u \to 1} \frac{u}{F(u)} \frac{Z_1}{u} \frac{F(z) dz}{z^2} . \]

It is easy to see that, if \((3.3)\) holds then we are allowed to substitute \((3.7)\) into \((3.5)\) on the way to the asymptotic relationship \((1.2)\):
\[ \bar{A}_r(u) = \frac{\gamma}{K_r} + \frac{\gamma c}{r} \frac{1}{\mu} \frac{1}{F(u)} \int_0^1 \frac{Z_1}{u} \frac{F(z) dz}{z^2} = \frac{\gamma}{r} \frac{b}{u} \frac{F(z) dz}{z}; \ u! 1 ; \]

But we have proved in Lemma 3.2 that \((3.3)\) is equivalent to the assertion that \((2.2)\) holds for some \(v > 1\), which is implied by the membership of \(F\) in \(A\). This proves Theorem 2.1 for the case where \(\gamma < c\).

If \(\gamma \geq c\), then, by the same argument as the proof of Lemma 4 in Kalashnikov & Konstantinides (2000), we can still obtain \((2.1)\). Hence, the validity of \((2.1)\) is independent of the safety loading condition. This ends the proof.

3.3 Proof of Theorem 2.2

From \((3.5)\), \((3.6)\) and noting that the functions \(\xi(z)\) and \(\bar{A}_0(z)\) are non-increasing in \(z \geq 0\), we derive the following lower bound of \(\bar{A}_r(u)\):
\[ \bar{A}_r(u) > \frac{\gamma}{K_r} + \frac{\gamma c}{r} \frac{1}{\mu} \frac{1}{F(u)} \int_0^1 \frac{Z_1}{u} \frac{F(z) dz}{z^2} = \frac{\gamma}{r} \frac{b}{u} \frac{F(z) dz}{z}; \ u! 1 ; \]

By these two bounds in \((3.8)\) and \((3.9)\), we get the proof of \((2.4)\).
3.4 Proof of Theorem 2.3

Note that, for our case, $F$ has a density function $f(x) = b \frac{1}{x^\alpha + B(x)}$ which is non-increasing in $x$, $0$ and tends to $0$ as $x \to 1$. Since (2.2) holds for some $v > 1$, then by Lemma 3.3 and Lemma 3.1, for any $v > 1$; there exists some $l(v) > 0$ such that (3.1) holds for all $u > l(v)$. The remaining proof of Theorem 2.3 is trivial.

4 Examples

Example 4.1. Now we put forward some concrete examples to illustrate how to determine the values of $l(v)$, $d(v)$, $\gamma(v)$, and $D(u)$ involved in our main results.

(1). Let $F$ be the Pareto distribution with $a > 1$; i.e. the tail of $F$ satisfies
\[ F(x) = \begin{cases} \frac{1}{x^\alpha} & \text{when } x > \cdot; \\ \cdot & \text{otherwise}; \end{cases} \quad \text{for } x > \cdot, \cdot > 0. \] (4.1)

Clearly, for any $v > 1$, we have
\begin{align*}
d(v) &= v^a; \\ \gamma(v) &= \frac{v^a i \cdot 1}{v^{a+1} i}; \quad \text{and } l(v) = \cdot.
\end{align*}

(2). Let $F$ be the Burr distribution with tail
\[ F(x) = \frac{\mu \cdot x^a}{\cdot + x^s}; \quad a; \cdot; s > 0; \quad x, 0. \] Modelling the proof of (3.2) with slight adjustment yields that, for any $v > 1$ and $u > 0$;
\begin{align*}
D(u), \frac{1}{v} 1 i \frac{\mu \cdot u^s}{\cdot + (vu)^s} &\geq \frac{v^a i \cdot 1}{v^{a+1}} > 0; \quad \text{as } u \to 1:
\end{align*}

(3). Let $F$ be the Weibull distribution with tail
\[ F(x) = \exp \left( \frac{\mu \cdot x^b}{\cdot + x^2} \right); \quad x > 0; \quad b > 0; \quad x > 0. \] (4.2)

We analogously obtain that, for any $v > 1$ and $u > 0$;
\begin{align*}
D(u), \frac{1}{v} 1 i \frac{\mu \cdot u^b}{\cdot + (vu)^2} &\geq \frac{v^a i \cdot 1}{v^{a+1}} > 0; \quad \text{as } u \to 1:
\end{align*}

Example 4.2. Next, we study the convergence rate of the deviation $i(u)$ to $0$ as $u \to 1$.

(1). Clearly, under the assumption that (2.2) holds for some $v > 1$, the inequality (2.4) implies
\[ i(u) = O \left( i u \cdot 1 + \gamma(u) + F(u) \cdot \right). \] (4.3)
(2). Recently Mikosch & Nagaev (2001) examined the case where the e.d.f. $F$ of claim size belongs to the class $D$ of d.f.’s with dominatedly varying tails. We say a d.f. $F$ supported on $[0; 1)$ belongs to the class $D$ if for some (or equivalently for any) $0 < c < 1$; it holds that

$$\limsup_{x \to 1} \frac{F(cx)}{F(x)} < 1.$$ 

The work in Mikosch & Nagaev (2001) indicates that if the e.d.f. $F$ belongs to $D$ with a finite mean, then $\xi(u) = O(u^{1/2})$ and therefore $\bar{\xi}(u) = O(u^{1/2})$ as $u \to 1$. From this and (4.3), we immediately obtain:

**Proposition.** In the classical model with a constant interest force $r$, if the e.d.f. $F$ of the claim size belongs to $A \setminus D$ and has a finite mean, then $\xi(u) = O(u^{1/2})$ and therefore $\bar{\xi}(u) = O(u^{1/2})$ as $u \to 1$.

We remark that the intersection $A \setminus D$ is a large subclass of heavy-tailed distributions. For example, if the e.d.f. $F$ belongs to $\text{ERV}(\hat{\beta}; \hat{\gamma})$ for $1 < \beta < \gamma < 1$; i.e.

$$v^{-1} \liminf_{x \to 1} \frac{F(vx)}{F(x)} \cdot \limsup_{x \to 1} \frac{F(vx)}{F(x)} \cdot v^\hat{\gamma}$$

for any $v > 1$; then $F$ satisfies all the conditions in Proposition. See Bingham et al. (1987) and Tang et al. (2001) for details. Clearly, if $\hat{\beta} = \hat{\gamma}$ then the class $\text{ERV}(\hat{\beta}; \hat{\gamma})$ coincides the class $R(\hat{\beta})$.

(3). We notice that in Proposition above the finite mean of the e.d.f. $F$ should be assumed in order for us to apply the related result in Mikosch & Nagaev (2001). So it excludes the case where $F$ follows a Pareto distribution with $0 < \alpha < 1$ (recall (4.1)), which is of interest in insurance practice. In this case, from the relation (1.17) of Kalashnikov & Tsitsiashvili (2000) and the concrete value of $s_F$ given in page 268 of that paper, we can obtain the asymptotic expression $\xi(u) = O(u^{1/2 - 1})$ as $u \to 1$. This formula, in combination with (4.3), yields $\xi(u) = O(u^{1/2 - 1})$ as $u \to 1$. This convergence rate is slightly worse than that in Proposition above;

(4). If $F$ follows the Weibull distribution given in (4.2), then, by the same approach and using the value of $s_F$ given in page 269 of Kalashnikov & Tsitsiashvili (2000), we obtain $\xi(u) = O(\ln u)^{1/3} u^{1/3}$ as $u \to 1$. This formula, together with (4.3), yields $\xi(u) = O(\ln u)^{1/3} u^{1/3}$ as $u \to 1$;

(5). If $F$ follows the Lognormal distribution with the density function

$$f(x) = \frac{1}{2\sqrt{\pi}} \exp \left(-\frac{\ln^2 x}{2}\right); \quad x > 0;$$

then, the same approach together with the value of $s_F$ given in page 270 of Kalashnikov & Tsitsiashvili (2000) gives $\xi(u) = O(\ln u)^{1/3} \exp\left(-\frac{2\ln u}{\sqrt{n}}\right)$ as $u \to 1$, from which we similarly obtain $\xi(u) = O(\ln u)^{1/3} \exp\left(-\frac{2\ln u}{\sqrt{n}}\right)$ as $u \to 1$. 11
5 Numerical Results

We write, for \( u > 0 \),
\[
\tilde{A}_t^i (u) = \max \left[ 1 + \frac{1}{r} \right] \left[ 1 + \frac{1}{r} \right] D(u) F(u); \quad 0 \quad \tilde{A}_t^o (u) = \frac{1}{r} D(u) F(u); \quad 0 \quad \tilde{A}_t^\xi (u) = \frac{1}{r} D(u) F(u);
\]
and
\[
\tilde{A}_t^\eta (u) = \frac{1}{r} D(u) F(u);
\]

Under the conditions of Theorem 2.1, we know from its Corollary that \( \tilde{A}_t^r (u) \approx \tilde{A}_t^\xi (u) \) as \( u \to 1 \); under the conditions of Theorem 2.2 we know that \( \tilde{A}_t^r (u) \left( \tilde{A}_t^o (u) \cdot \tilde{A}_t^\eta (u) \cdot \tilde{A}_t^\eta (u) \right) \) for \( u > 0 \). We proceed to the calculation of these estimates in the Pareto and Weibull cases. The upper bounds for the ruin probability \( \tilde{A}_0(u) \) involved can be found in Kalashnikov & Tsitsiashvili (2000). We take the numerical results produced in the package Mathematica. In each table we vary the interest force from 0:01 to 0:31 by step 0:1. For simplicity we assume \( c = 1 \).

In the first case we assume the e.d.f. \( F \) follows a Pareto distribution, the tail of which has form
\[
F(u) = (1 + bu)^{-a}; \quad a; b; u > 0;
\]

We take the parameters above with values \( a = 3 \), \( b = 0:5 \). In the tables from 1 to 3 we take the numerical results for \( \frac{1}{r} = 0:1 \) and \( u = 9, 100, 1000 \).

**Table 1.** Pareto case with \( a = 3 \), \( b = 0:5 \), \( \frac{1}{r} = 0:1 \), \( u = 9 \), \( \tilde{A}_0(u) = 0:00725 \), \( \xi (u) = 0:007617 \), \( D(u) = 0:718924 \).

<table>
<thead>
<tr>
<th>( r )</th>
<th>( \tilde{A}_0^o (u) )</th>
<th>( \tilde{A}_t^o (u) )</th>
<th>( \tilde{A}_t^\xi (u) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>0.00480123</td>
<td>0.00070027</td>
<td>0.000116647</td>
</tr>
<tr>
<td>0.11</td>
<td>0.000435647</td>
<td>0.000116647</td>
<td>0.000574674</td>
</tr>
<tr>
<td>0.21</td>
<td>0.00022863</td>
<td>0.000110868</td>
<td>0.000287031</td>
</tr>
<tr>
<td>0.31</td>
<td>0.000154878</td>
<td>0.0000928057</td>
<td>0.000189465</td>
</tr>
</tbody>
</table>

**Table 2.** Pareto case with \( a = 3 \), \( b = 0:5 \), \( \frac{1}{r} = 0:1 \), \( u = 100 \), \( \tilde{A}_0(u) = 0:0855969 \), \( \xi (u) = 0:007617 \), \( D(u) = 0:718924 \).

<table>
<thead>
<tr>
<th>( r )</th>
<th>( \tilde{A}_0^o (u) )</th>
<th>( \tilde{A}_t^o (u) )</th>
<th>( \tilde{A}_t^\xi (u) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>5.63147^2 10^{-7}</td>
<td>1.84765^2 10^{-7}</td>
<td>6.64245^2 10^{-7}</td>
</tr>
<tr>
<td>0.11</td>
<td>5.11952^2 10^{-8}</td>
<td>4.5352^2 10^{-8}</td>
<td>5.3162^2 10^{-8}</td>
</tr>
<tr>
<td>0.21</td>
<td>2.68165^2 10^{-8}</td>
<td>2.51156^2 10^{-8}</td>
<td>2.75028^2 10^{-8}</td>
</tr>
<tr>
<td>0.31</td>
<td>1.8166^2 10^{-8}</td>
<td>1.73592^2 10^{-8}</td>
<td>1.85435^2 10^{-8}</td>
</tr>
</tbody>
</table>

**Table 3.** Pareto case with \( a = 3 \), \( b = 0:5 \), \( \frac{1}{r} = 0:1 \), \( u = 1000 \), \( \tilde{A}_0(u) = 8:85^2 10^{-10} \), \( \xi (u) = 0:00161071 \), \( D(u) = 0:749592 \).

12
<table>
<thead>
<tr>
<th>r</th>
<th>$\tilde{A}_v^2(u)$</th>
<th>$\tilde{A}_v^1(u)$</th>
<th>$\tilde{A}_v^2(u)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>5.96092 $10^{-11}$</td>
<td>5.23476 $10^{-11}$</td>
<td>6.15473 $10^{-11}$</td>
</tr>
<tr>
<td>0.11</td>
<td>5.4192 $10^{-12}$</td>
<td>5.35095 $10^{-12}$</td>
<td>5.44695 $10^{-12}$</td>
</tr>
<tr>
<td>0.21</td>
<td>2.83852 $10^{-12}$</td>
<td>2.81905 $10^{-12}$</td>
<td>2.84911 $10^{-12}$</td>
</tr>
<tr>
<td>0.31</td>
<td>1.92287 $10^{-12}$</td>
<td>1.91359 $10^{-12}$</td>
<td>1.92907 $10^{-12}$</td>
</tr>
</tbody>
</table>

In the tables from 4 to 6 we take $1/2=0.9$ and $u=9,100,1000$.

Table 4. Pareto-like case with $a = 3$, $b = 0.5$, $1/2=0.9$, $u=9$, $\tilde{A}_v(u) = 0.356$, $\zeta(u) = 5.58106$, $D(u) = 0.718924$.

<table>
<thead>
<tr>
<th>r</th>
<th>$\tilde{A}_v^2(u)$</th>
<th>$\tilde{A}_v^1(u)$</th>
<th>$\tilde{A}_v^2(u)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>0.0432111</td>
<td>0</td>
<td>0.612822</td>
</tr>
<tr>
<td>0.11</td>
<td>0.0392828</td>
<td>0</td>
<td>0.0550738</td>
</tr>
<tr>
<td>0.21</td>
<td>0.0205767</td>
<td>0</td>
<td>0.0287223</td>
</tr>
<tr>
<td>0.31</td>
<td>0.013939</td>
<td>0</td>
<td>0.0194123</td>
</tr>
</tbody>
</table>

Table 5. Pareto case with $a = 3$, $b = 0.5$, $1/2=0.9$, $u=100$, $\tilde{A}_v(u) = 0.000348$, $\zeta(u) = 4.12917$, $D(u) = 0.74702$.

<table>
<thead>
<tr>
<th>r</th>
<th>$\tilde{A}_v^2(u)$</th>
<th>$\tilde{A}_v^1(u)$</th>
<th>$\tilde{A}_v^2(u)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>5.06832 $10^{-6}$</td>
<td>0</td>
<td>0.0000339539</td>
</tr>
<tr>
<td>0.11</td>
<td>4.60756 $10^{-7}$</td>
<td>0</td>
<td>3.0217 $10^{-6}$</td>
</tr>
<tr>
<td>0.21</td>
<td>2.41349 $10^{-7}$</td>
<td>0</td>
<td>1.5797 $10^{-6}$</td>
</tr>
<tr>
<td>0.31</td>
<td>1.63494 $10^{-7}$</td>
<td>0</td>
<td>1.06933 $10^{-6}$</td>
</tr>
</tbody>
</table>

Table 6. Pareto case with $a = 3$, $b = 0.5$, $1/2=0.9$, $u=1000$, $\tilde{A}_v(u) = 8.242 \times 10^{-8}$, $\zeta(u) = 0.151325$, $D(u) = 0.749592$.

<table>
<thead>
<tr>
<th>r</th>
<th>$\tilde{A}_v^2(u)$</th>
<th>$\tilde{A}_v^1(u)$</th>
<th>$\tilde{A}_v^2(u)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>5.36481 $10^{-10}$</td>
<td>4.44297 $10^{-10}$</td>
<td>6.61076 $10^{-10}$</td>
</tr>
<tr>
<td>0.11</td>
<td>4.8771 $10^{-11}$</td>
<td>4.57194 $10^{-11}$</td>
<td>5.87635 $10^{-11}$</td>
</tr>
<tr>
<td>0.21</td>
<td>2.55467 $10^{-11}$</td>
<td>2.40937 $10^{-11}$</td>
<td>3.07444 $10^{-11}$</td>
</tr>
<tr>
<td>0.31</td>
<td>1.73058 $10^{-11}$</td>
<td>1.63568 $10^{-11}$</td>
<td>2.08181 $10^{-11}$</td>
</tr>
</tbody>
</table>

Now we assume $a = 5$, $b = 0.25$. In the tables from 7 to 9 we take $1/2=0.1$ and $u=9,100,1000$.

Table 7. Pareto case with $a = 5$, $b = 0.25$, $1/2=0.1$, $u=9$, $\tilde{A}_v(u) = 0.000355$, $\zeta(u) = 0.158478$, $D(u) = 0.78556$.

<table>
<thead>
<tr>
<th>r</th>
<th>$\tilde{A}_v^2(u)$</th>
<th>$\tilde{A}_v^1(u)$</th>
<th>$\tilde{A}_v^2(u)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>0.00240724</td>
<td>0</td>
<td>0.003497</td>
</tr>
<tr>
<td>0.11</td>
<td>0.00021884</td>
<td>0.0000693594</td>
<td>0.000293123</td>
</tr>
<tr>
<td>0.21</td>
<td>0.000114631</td>
<td>0.0000591667</td>
<td>0.000148644</td>
</tr>
<tr>
<td>0.31</td>
<td>0.0000776531</td>
<td>0.0000482031</td>
<td>0.0000989523</td>
</tr>
</tbody>
</table>
Table 8. Pareto case with \( a = 5, b = 0.25, \frac{1}{2} = 0.1, u = 100, \tilde{A}_0(u) = 9.472 \times 10^9, \zeta(u) = 0.0126497, D(u) = 0.828618.

<table>
<thead>
<tr>
<th>( r )</th>
<th>( \tilde{A}_i^0(u) )</th>
<th>( \tilde{A}_i^1(u) )</th>
<th>( \tilde{A}_i^+(u) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>6.974092 \times 10^9</td>
<td>2.747572 \times 10^9</td>
<td>7.801782 \times 10^9</td>
</tr>
<tr>
<td>0.11</td>
<td>6.340882 \times 10^{10}</td>
<td>5.685882 \times 10^{10}</td>
<td>6.546142 \times 10^{10}</td>
</tr>
<tr>
<td>0.21</td>
<td>3.320992 \times 10^{10}</td>
<td>3.130132 \times 10^{10}</td>
<td>3.402912 \times 10^{10}</td>
</tr>
<tr>
<td>0.31</td>
<td>2.249712 \times 10^{10}</td>
<td>2.158982 \times 10^{10}</td>
<td>2.298592 \times 10^{10}</td>
</tr>
</tbody>
</table>

Table 9. Pareto case with \( a = 5, b = 0.25, \frac{1}{2} = 0.1, u = 1000, \tilde{A}_0(u) = 1.122 \times 10^{13}, \zeta(u) = 0.00422063, D(u) = 1.

<table>
<thead>
<tr>
<th>( r )</th>
<th>( \tilde{A}_i^0(u) )</th>
<th>( \tilde{A}_i^1(u) )</th>
<th>( \tilde{A}_i^+(u) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>1.003762 \times 10^{14}</td>
<td>9.125122 \times 10^{15}</td>
<td>1.00822 \times 10^{14}</td>
</tr>
<tr>
<td>0.11</td>
<td>9.125122 \times 10^{16}</td>
<td>9.042912 \times 10^{16}</td>
<td>9.163642 \times 10^{16}</td>
</tr>
<tr>
<td>0.21</td>
<td>4.779832 \times 10^{16}</td>
<td>4.757172 \times 10^{16}</td>
<td>4.8210^{16}</td>
</tr>
<tr>
<td>0.31</td>
<td>3.237952 \times 10^{16}</td>
<td>3.227542 \times 10^{16}</td>
<td>3.251612 \times 10^{16}</td>
</tr>
</tbody>
</table>

In the tables from 10 to 12 we take \( \frac{1}{2} = 0.9 \) and \( u = 9, 100, 1000 \).

Table 10. Pareto case with \( a = 5, b = 0.25, \frac{1}{2} = 0.9, u = 9, \tilde{A}_0(u) = 0.364, \zeta(u) = 13:6648, D(u) = 0.78556.

<table>
<thead>
<tr>
<th>( r )</th>
<th>( \tilde{A}_i^0(u) )</th>
<th>( \tilde{A}_i^1(u) )</th>
<th>( \tilde{A}_i^+(u) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>0.0216652</td>
<td>0</td>
<td>0.635431</td>
</tr>
<tr>
<td>0.11</td>
<td>0.00196956</td>
<td>0</td>
<td>0.0575433</td>
</tr>
<tr>
<td>0.21</td>
<td>0.00103168</td>
<td>0</td>
<td>0.0300977</td>
</tr>
<tr>
<td>0.31</td>
<td>0.000698878</td>
<td>0</td>
<td>0.0203731</td>
</tr>
</tbody>
</table>

Table 11. Pareto case with \( a = 5, b = 0.25, \frac{1}{2} = 0.9, u = 100, \tilde{A}_0(u) = 0.00013, \zeta(u) = 170:62, D(u) = 0.828618:

<table>
<thead>
<tr>
<th>( r )</th>
<th>( \tilde{A}_i^0(u) )</th>
<th>( \tilde{A}_i^1(u) )</th>
<th>( \tilde{A}_i^+(u) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>6.276682 \times 10^8</td>
<td>0</td>
<td>0.0000129952</td>
</tr>
<tr>
<td>0.11</td>
<td>5.706072 \times 10^8</td>
<td>0</td>
<td>1.180892 \times 10^6</td>
</tr>
<tr>
<td>0.21</td>
<td>2.988892 \times 10^8</td>
<td>0</td>
<td>6.185382 \times 10^7</td>
</tr>
<tr>
<td>0.31</td>
<td>2.024742 \times 10^8</td>
<td>0</td>
<td>4.190042 \times 10^7</td>
</tr>
</tbody>
</table>

Table 12. Pareto case with \( a = 5, b = 0.25, \frac{1}{2} = 0.9, u = 1000, \tilde{A}_0(u) = 1.142 \times 10^{11}, \zeta(u) = 0.261917, D(u) = 1.

<table>
<thead>
<tr>
<th>( r )</th>
<th>( \tilde{A}_i^0(u) )</th>
<th>( \tilde{A}_i^1(u) )</th>
<th>( \tilde{A}_i^+(u) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>9.033872 \times 10^{14}</td>
<td>8.212612 \times 10^{14}</td>
<td>1.142 \times 10^{13}</td>
</tr>
<tr>
<td>0.11</td>
<td>8.212612 \times 10^{15}</td>
<td>8.138622 \times 10^{15}</td>
<td>1.036362 \times 10^{14}</td>
</tr>
<tr>
<td>0.21</td>
<td>4.301842 \times 10^{15}</td>
<td>4.281462 \times 10^{15}</td>
<td>5.428572 \times 10^{15}</td>
</tr>
<tr>
<td>0.31</td>
<td>2.914152 \times 10^{15}</td>
<td>2.904782 \times 10^{15}</td>
<td>3.677422 \times 10^{15}</td>
</tr>
</tbody>
</table>
In the second case we assume the e.d.f. \( F \) follows a Weibull distribution with the form (4.2). We take the parameters in (4.2) with values, \( \lambda = 4.52874, b = 0.1 \).

In the tables from 13 to 16 we take \( \lambda = 0.5 \) and \( u = 10, 100, 1000, 10000 \).

Table 13. Weibull case with \( \lambda = 4.52874, b = 0.1, \lambda = 0.5, u = 10, \tilde{A}_0(u) = 0.00335, \zeta(u) = 0.00255419, D(u) = 0.377721:

<table>
<thead>
<tr>
<th>( r )</th>
<th>( \tilde{A}_0^0(u) )</th>
<th>( \tilde{A}_0^1(u) )</th>
<th>( \tilde{A}_0^2(u) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>0.00631071</td>
<td>0</td>
<td>0.0158612</td>
</tr>
<tr>
<td>0.11</td>
<td>0.000573701</td>
<td>0</td>
<td>0.00103277</td>
</tr>
<tr>
<td>0.21</td>
<td>0.00030051</td>
<td>0.000040936</td>
<td>0.000464925</td>
</tr>
<tr>
<td>0.31</td>
<td>0.000203571</td>
<td>0.0000701342</td>
<td>0.000288563</td>
</tr>
</tbody>
</table>

Table 14. Weibull case with \( \lambda = 4.52874, b = 0.1, \lambda = 0.5, u = 100, \tilde{A}_0(u) = 0.000766, \zeta(u) = 0.00324607, D(u) = 0.431774:

<table>
<thead>
<tr>
<th>( r )</th>
<th>( \tilde{A}_0^0(u) )</th>
<th>( \tilde{A}_0^1(u) )</th>
<th>( \tilde{A}_0^2(u) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>0.000164834</td>
<td>0</td>
<td>0.00027483</td>
</tr>
<tr>
<td>0.11</td>
<td>0.000149849</td>
<td>0.0000120136</td>
<td>0.0000167677</td>
</tr>
<tr>
<td>0.21</td>
<td>7.849252(10^{-6})</td>
<td>6.981462(10^{-6})</td>
<td>8.391782(10^{-6})</td>
</tr>
<tr>
<td>0.31</td>
<td>5.317242(10^{-6})</td>
<td>4.9043(10^{-6})</td>
<td>5.585362(10^{-6})</td>
</tr>
</tbody>
</table>

Table 15. Weibull case with \( \lambda = 4.52874, b = 0.1, \lambda = 0.5, u = 1000, \tilde{A}_0(u) = 0.00012, \zeta(u) = 0.00803757, D(u) = 0.487477:

<table>
<thead>
<tr>
<th>( r )</th>
<th>( \tilde{A}_0^0(u) )</th>
<th>( \tilde{A}_0^1(u) )</th>
<th>( \tilde{A}_0^2(u) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>2.901542(10^{-6})</td>
<td>2.335552(10^{-6})</td>
<td>3.227432(10^{-6})</td>
</tr>
<tr>
<td>0.11</td>
<td>2.637772(10^{-7})</td>
<td>2.566392(10^{-7})</td>
<td>2.70692(10^{-7})</td>
</tr>
<tr>
<td>0.21</td>
<td>1.381692(10^{-7})</td>
<td>1.35642(10^{-7})</td>
<td>1.41172(10^{-7})</td>
</tr>
<tr>
<td>0.31</td>
<td>9.359812(10^{-8})</td>
<td>9.217792(10^{-8})</td>
<td>9.54812(10^{-8})</td>
</tr>
</tbody>
</table>

Table 16. Weibull case with \( \lambda = 4.52874, b = 0.1, \lambda = 0.5, u = 10000, \tilde{A}_0(u) = 0.0000115, \zeta(u) = 0.00252021, D(u) = 0.543501:

<table>
<thead>
<tr>
<th>( r )</th>
<th>( \tilde{A}_0^0(u) )</th>
<th>( \tilde{A}_0^1(u) )</th>
<th>( \tilde{A}_0^2(u) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>3.117272(10^{-8})</td>
<td>3.053892(10^{-8})</td>
<td>3.157712(10^{-8})</td>
</tr>
<tr>
<td>0.11</td>
<td>2.833882(10^{-9})</td>
<td>2.823122(10^{-9})</td>
<td>2.849252(10^{-9})</td>
</tr>
<tr>
<td>0.21</td>
<td>1.484422(10^{-9})</td>
<td>1.479962(10^{-9})</td>
<td>1.491922(10^{-9})</td>
</tr>
<tr>
<td>0.31</td>
<td>1.005572(10^{-9})</td>
<td>1.002842(10^{-9})</td>
<td>1.010532(10^{-9})</td>
</tr>
</tbody>
</table>

In the tables from 17 to 20 we take \( \lambda = 0.95 \) and \( u \) equal to 10, 100, 1000, 10000.

Table 17. Weibull case with \( \lambda = 4.52874, b = 0.1, \lambda = 0.95, u = 10, \tilde{A}_0(u) = 0.0658, \zeta(u) = 0.0364189, D(u) = 0.377721. |
Table 18. Weibull case with $\gamma = 4.52874$, $b = 0.1$, $\lambda = 0.95$, $u = 100$, $\bar{A}_0(u) = 0.0156$, $\bar{c}(u) = 0.0753496$, $D(u) = 0.431774$.

<table>
<thead>
<tr>
<th>r</th>
<th>$\bar{A}_0(u)$</th>
<th>$\bar{A}_1(u)$</th>
<th>$\bar{A}_2(u)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>0.0119903</td>
<td>0</td>
<td>0.0334215</td>
</tr>
<tr>
<td>0.11</td>
<td>0.00109003</td>
<td>0</td>
<td>0.00226093</td>
</tr>
<tr>
<td>0.21</td>
<td>0.000570969</td>
<td>0</td>
<td>0.0010398</td>
</tr>
<tr>
<td>0.31</td>
<td>0.000386785</td>
<td>0.0000673069</td>
<td>0.00654247</td>
</tr>
</tbody>
</table>

Table 19. Weibull case with $\gamma = 4.52874$, $b = 0.1$, $\lambda = 0.95$, $u = 1000$, $\bar{A}_0(u) = 0.00246$, $\bar{c}(u) = 0.0876195$, $D(u) = 0.487477$.

<table>
<thead>
<tr>
<th>r</th>
<th>$\bar{A}_0(u)$</th>
<th>$\bar{A}_1(u)$</th>
<th>$\bar{A}_2(u)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>5.512932 10^{-6}</td>
<td>3.961432 10^{-6}</td>
<td>7.061092 10^{-6}</td>
</tr>
<tr>
<td>0.11</td>
<td>5.011752 10^{-7}</td>
<td>4.443312 10^{-7}</td>
<td>5.98762 10^{-7}</td>
</tr>
<tr>
<td>0.21</td>
<td>2.62522 10^{-7}</td>
<td>2.350442 10^{-7}</td>
<td>3.12459 10^{-7}</td>
</tr>
<tr>
<td>0.31</td>
<td>1.77836 10^{-7}</td>
<td>1.597792 10^{-7}</td>
<td>2.11381 10^{-7}</td>
</tr>
</tbody>
</table>

Table 20. Weibull case with $\gamma = 4.52874$, $b = 0.1$, $\lambda = 0.95$, $u = 10000$, $\bar{A}_0(u) = 0.00024$, $\bar{c}(u) = 0.101166$, $D(u) = 0.543501$.

<table>
<thead>
<tr>
<th>r</th>
<th>$\bar{A}_0(u)$</th>
<th>$\bar{A}_1(u)$</th>
<th>$\bar{A}_2(u)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>5.92282 10^{-8}</td>
<td>5.31033 10^{-8}</td>
<td>7.07742 10^{-8}</td>
</tr>
<tr>
<td>0.11</td>
<td>5.38438 10^{-9}</td>
<td>4.91666 10^{-9}</td>
<td>6.39334 10^{-9}</td>
</tr>
<tr>
<td>0.21</td>
<td>2.82039 10^{-9}</td>
<td>2.577642 10^{-9}</td>
<td>3.34787 10^{-9}</td>
</tr>
<tr>
<td>0.31</td>
<td>1.91059 10^{-9}</td>
<td>1.74668 10^{-9}</td>
<td>2.26767 10^{-9}</td>
</tr>
</tbody>
</table>

Acknowledgment. We feel the duty to express our deep gratitude to the late Prof. V.V. Kalashnikov for his numerous valuable discussions on this paper. We would like to thank the anonymous referee for the constructive suggestions and comments on the previous versions of this paper.
References


Kalashnikov, V.V., Tsitsiashvili, G.Sh., 2000. Tight approximation of basic characteristics of classical and non-classical surplus process. ARCH 00V210(2000-9) 2, 251-293.


