Optimal Reinsurance Programs

An optimal Combination of several Reinsurance Protections on an heterogeneous Insurance Portfolio

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Abstract

In most practical cases the reinsurance protection of an insurance portfolio is not limited to one reinsurance type such as quota share, surplus, excess of loss or stop loss, but is organised through a combination of several methods of protection, a so-called reinsurance program.

In this paper we will analyze optimal reinsurance programs for a given portfolio based on the “mean – variance” optimisation criterion.

Special attention is given to a description of a “surplus reinsurance” in combination with an “excess of loss per risk protection” for an heterogeneous insurance portfolio.

We derive the equations one needs to solve for finding the optimal solution for the following combinations: excess of loss after surplus, excess of loss after quota share, stop loss after quota share, quota share after stop loss, quota share after excess of loss, quota share before surplus and quota share after surplus.

It turns out that the application order of the reinsurance protections has its importance.

Keywords

Proportional & non-proportional reinsurance; optimal reinsurance, mean variance criterion.

1 Introduction

In the actuarial literature most papers concerning optimal reinsurance are limited to one reinsurance protection given an optimisation criterion. Exceptions are Centeno (1985, 2002), Kaluszka (2001) and Schmitter (2001) who consider optimal solutions for a quota share - excess of loss combination.

In practice reinsurance however is much more complicated. Even for a relatively simple property portfolio one will construct reinsurance based on a combination of different types of protections. If we look in detail to a more or less traditional reinsurance construction one will detect that, among others,

• A combination of proportional and non-proportional reinsurance protections is frequently used.
• Many times the proportional reinsurance will include (or only exist of) a surplus protection. However the optimal proportional reinsurance literature is essentially limited to quota share protections (a link to surplus reinsurance exists via a variable quota share, but this is definitely not the same).
• There exists a combination of protections such as a "per risk" and "per event" protection (for example: conflagration, storm, earthquake, …) or a personal accident program. If both types of protections are independent of each other (in the sense of the protection per risk will not influence the protection per event), then one can describe the reinsurance by independent reinsurance protections (one for the per risk and one for the per event protection). Otherwise one needs a much more complicated model to describe the risk and the interactive reinsurance protections.
• Non-proportional protections as well as surplus reinsurance are mostly subdivided in several layers. Each layer will have its own conditions. These conditions will result to a sum of successive layers which is not equivalent to one layer (for example: the first layer will include an additional franchise (= Annual Aggregate Deductible = AAD); the different layers will include different co-reinsurance clauses; each layer will include a different number of reinstatements; each layer will include other types of reinstatement clauses (pro rata claim; pro rata time; pro rata claim & time); each layer will include a different no-claim bonus; …).
• In non-proportional reinsurance one will cede many times the risk up to a certain level (= upper limit). The risk exceeding this upper limit goes back to the retention of the cedent. The same applies to the risk after exhaustion of the limited number of reinstatements per layer.

Note that this overflow risk will depend on the first retained risk, which makes the analyses of the retention much more complicated.

However, if one treats the optimal reinsurance question, one will (nearly) never take this overflow risk
into account. Nevertheless it is clear that for some extreme risks (for example: wind storm, earthquake, …) this overflow risk can strongly influence the analysis of the retained risk.

- Most portfolios will include a lot of heterogeneity. This heterogeneity will imply an additional deviation risk one likes to cede to the reinsurance market. Many times this heterogeneity is an important reason for combining several reinsurance protections. A classical example is the heterogeneity induced by the difference in sum insured per policy, but many other differences between policies will induce differences in exposure (for example: difference in coverage (theft, third party protection, …), difference in type (simple risks - commercial risks - industrial risks - …)).

In this paper we will not tackle all these problems but we will concentrate on 3 items:
- Describing the combination of basic reinsurance protections. Basic stands here for: no subdivision in layers, no subdivision of the principal protection in subcontracts at different conditions, no overflow risk due to reinsurance limitations, ….
- Analysing and characterising the influence of the heterogeneity induced by the difference in risk exposure per policy (for example sum insured).
- Describing, discussing and applying of a practical optimisation criterion for reinsurance.

We will not deal explicitly with the typical long-tail risk items, but limit ourselves essentially to the so-called short-tail business (for example property, personal accident, …). However, if one ignores the time between the loss occurrence (or the claims made date) and the loss payments, one can drop that limitation.

Classical reinsurance can be classified in 2 types: the proportional and the non-proportional (for a detailed description see for instance Carter (1995); Geratwohl (1980, 1992) and Kiln (1982)). To introduce some notations and to summarise the main influence of these 2 methods, we consider a portfolio of \( n \) risks. Each risk \( i \) is characterised by an entity or risk \( R_i \), risk exposure \( K_i \) (for example the sum insured value \( C_i \) or some Maximum Possible Loss value, in general all terms of the original policy), premium \( P_i \), individual claims \( S_{i,j} \) and claims per year & per entity \( S_i = \sum_{j=1}^{n_j} S_{i,j} \) for \( i = 1, .., n \) and \( j = 1, .., n_i \).

1.1 Proportional reinsurance

All the characteristics of each risk (= risk exposure, premium, claim) are proportionally shared between cedent and reinsurer (for example: claim potential, risk of wrong tariff or conditions, timing risk, ). If we consider a proportional cession per risk equal to \( 1 - a_i \in [0,1] \) and consequentially a retention per risk equal to \( a_i \), we obtain:

<table>
<thead>
<tr>
<th></th>
<th>Retention</th>
<th>Proportional cession</th>
</tr>
</thead>
<tbody>
<tr>
<td>Premium</td>
<td>[ \sum_{i=1}^{n} a_i \cdot P_i ]</td>
<td>[ \sum_{i=1}^{n} (1 - a_i) \cdot P_i ]</td>
</tr>
<tr>
<td>Exposure</td>
<td>[ \sum_{i=1}^{n} a_i \cdot K_i ]</td>
<td>[ \sum_{i=1}^{n} (1 - a_i) \cdot K_i ]</td>
</tr>
<tr>
<td>Claim</td>
<td>[ \sum_{i=1}^{n} \left( \sum_{j=1}^{n_j} a_i \cdot S_{i,j} \right) = \sum_{i=1}^{n} a_i \cdot S_i ]</td>
<td>[ \sum_{i=1}^{n} \left( \sum_{j=1}^{n_j} (1 - a_i) \cdot S_{i,j} \right) = \sum_{i=1}^{n} (1 - a_i) \cdot S_i ]</td>
</tr>
</tbody>
</table>

In practice one will use simple rules to define these cession proportions. Basically one uses two rules, if one makes abstraction from the less essential differences, which will lead to a "surplus" and a "quota share" type of reinsurance.
**For a surplus**, one uses the non-negative real valued exposure function $K : i \rightarrow K(i) = K_i \geq 0$ to distribute proportionally the exposure to the surplus reinsurers in such a way that

$$1 - a_i = \frac{(K_i - \text{line})_+}{K_i} = \begin{cases} 0 & \text{if } \text{line} \geq K_i \\ \frac{(K_i - \text{line})}{K_i} & \text{if } \text{line} < K_i \end{cases},$$

with the so-called “line” the amount of retained exposure by the cedent.

**For a quota share** the cedent cedes a fixed proportion for all risks, this means that $a_i = a$ for all $i$.

### 1.2 Non-proportional reinsurance

With this reinsurance method one tries to control the consequences of the portfolio by a direct attack on the claim amounts. The reinsurer will take over the claim amounts that (in some sense) are too high. In practice this means that the reinsurer takes for its account the claim amount that exceeds a franchise or priority.

<table>
<thead>
<tr>
<th>Retention</th>
<th>Non-proportional cession</th>
</tr>
</thead>
<tbody>
<tr>
<td>For each claim $S$</td>
<td>Minimum of $(S, \text{priority})$</td>
</tr>
</tbody>
</table>

In practice one has different ways to define the “level” of a claim amount:

- **Per risk**: $S = (S_i, \text{priority})$, which corresponds to the so-called excess of loss per risk reinsurance.

- **Per accident / event**: $S = (S_i - \text{priority})$, with $S_g = \sum_{i,j \in g} S_{i,j}$ if the claim $j$ of the risk $i$ is the consequence of the event/accident $g$, which is the so-called excess of loss per event/accident reinsurance.

- **Per year**: $S = \left[ \left( \sum_{i=1}^{n} \sum_{j=1}^{n_i} S_{i,j} \right) - \text{priority} \right]_+$, which defines the so-called stop loss reinsurance.

For a non-proportional cover the lot between cedent and reinsurer is (strongly) separated: good or bad results for one party do not imply necessarily good or bad results for the other party implying that both results are only correlated on a limited basis.

In non-proportional covers the intervention of the reinsurer is often limited to a special part of the risk (e.g. limited to well defined events such as catastrophe risks like windstorm, earthquake, flood, …, limited cover such as reinstatements, upper limit, …, timing of recovery such as after exhaustion of the priority, …, exclusion and so on), which will more strongly separate the results between cedent and reinsurer.

### 1.3 Combination of reinsurance protections in practice

To combine non-proportional reinsurance in a logical way, one needs to respect following order: first an excess of loss per risk, secondly an excess of loss per event or accident and finally a stop loss (if one goes the other way around, one can not assign the recuperation of the $n$-th protection to the $n-1$-th without any supplemental allocation rule).

A quota share can be ceded in any order, because the fixed proportion is always applicable to the portfolio as a whole and / or after any reinsurance protection.

If one separates a specific part of the insured perils then it is possible to construct two (or more) parallel reinsurance protections: one for the separated insured dangers and another one reinsuring all the remaining dangers. In practice this is sometimes done for the natural catastrophic events, because they are almost exclusively protected by excess of loss per event covers.

A surplus protection should be the first in the series of the protection, if applicable, but as stated before a quota share can always precede.
Note that a surplus protection after an excess of loss per risk will imply that the higher sum insured risks, which are systematically ceded to the surplus, have much more potential to recover from an excess intervention. So, one should recover the excess premium, but (much) more than on a pro rata basis from the surplus reinsurers. In practice one will nearly always avoid such a difficulty. A surplus after an excess of loss per event is logically impossible, because the recovery from the excess is global and cannot be correctly distributed over the individual risks. However, as stated before, it is possible to isolate some special events from the surplus cession (for example: windstorm and earthquake risks).

A typical reinsurance structure for property business is given in Table 1. One should note that a quota share normally once appears in the series of protection, a multiple occurrence is rare but not excluded.

### 2 The optimisation criterion and reinsurance loading structure

Several optimisation criteria exist for reinsurance in the literature. Best known is the "Mean - Variance criterion", which concerns the maximisation of the expected profit under the constraint of a fixed variance, or equivalent, the minimisation of the variance under the constraint of a fixed profit. Another popular method is the “Probability of ruin criterion”, which concerns the maximisation of profit under the constraint of ruin probability less then a fixed value, or equivalent, minimisation of the ruin probability under the constraint of a minimal expected profit.

Note that the finite or the infinite ruin probability can be chosen as a criterion but also the time model (discrete or continuous) will influence the solution. Sometimes the maximisation of the adjustment coefficient is used as the optimisation criterion due to the fact that this coefficient can be used to define an upper bound for the ruin probability with infinite time horizon. Furthermore also the dividend policy and the utility stabilisation criterion are described in the literature. The first one is in some sense a generalisation of the ruin criterion and the second one of the mean variance principle.


In this paper we will use the mean-variance optimisation criterion. However we will introduce this criterion in a slightly different form, with the advantage of a more useful and more general interpretation.

We will minimise the cost of the net retained risk (= retained risk after intervention of the whole reinsurance program). The cost of retaining the risk will be defined as: the total of the reinsurance loading i.e. additional price to be paid above the expected risk ceded to the reinsurers plus a fixed multiple of the retained standard deviation.

We assume that the reinsurance loading is proportional with the expected risk ceded to the reinsurer, which corresponds to the expected value principle of (reinsurance) premium calculation. This proportion will vary per type of protection and can depend on the sequence order of the reinsurance protection.

The multiple of the retained standard deviation is interpreted as the cost of capital allocation to carry the risk. We assume that the allocated capital is proportional to the standard deviation (for example a factor 4.5), but due to compensation with other retained risks reduced with some factor (for example 0.80). This allocated capital should give a certain return (for example 25%). This implies an allocated capital cost that is also proportional to the retained standard deviation (for example 4.5 x 0.80 x 0.25 = 0.90).

(Note that the assumption of an explicit link between standard deviation and allocated capital implies that we restrict ourselves to risks with bounded volatility).

Important to note is that introducing this standard deviation constraint, implies that the cost of capital allocation as a multiple of the standard deviation is not necessary to find an optimal reinsurance solution. This observation plus the fact that working with standard deviation or variation constraints is equivalent, implies that our "mean-variance criterion” drops down to the classical "mean-variance criterion". 

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Formal description of the criterion and loading structure:

- **Premium income Cedent =** $PI$ (net of facultative reinsurance). Note that we don't suppose an explicit link between $S$ and $PI$, but in general $PI > E(S)$, with $E()$ the abbreviation for expected value.

- **Retained risk =** $R(S)$. This is the risk retained by the cedent after application of all the reinsurance protections.

- **Allocated capital cost for the cedent =** $\alpha \cdot Std[R(S)]$, with $Std$ the abbreviation for standard deviation. The parameter $\alpha (\geq 0)$ is supposed to be fixed.

- **Reinsured or ceded risk =** $\sum_{i=1}^{n} T_i(S)$ with $T_i$ one of the reinsurance protections (i.e. quota share, surplus, stop loss or excess of loss).

- **Reinsurance Loading cost =** $\sum_{i=1}^{n} \lambda_i \cdot E[T_i(S)]$, with $\lambda_i (\geq 0)$ fixed. In general the loading factors $\lambda_i$ can be subdivided in 2 parts. A first component really paid to the reinsurer (to finance his costs and gross margin) and a second part as a (direct and indirect) cost to be borne by the cedent for managing and organising the cessions of the reinsurance treaties.

- **Note that following relations hold:**

$$
S = R(S) + \sum_{i=1}^{n} T_i(S), \text{ implying } E[S] = E[R(S)] + E\left[\sum_{i=1}^{n} T_i(S)\right]
$$

$$
Std[S] \leq Std[R(S)] + Std\left[\sum_{i=1}^{n} T_i(S)\right].
$$

We define the (gross) profit $G_i$ of the cedent's (sub-) portfolio after reinsurance by "**Premium Income minus the retained risk minus the reinsurance cost minus the retained risk allocated capital cost**" or

$$
G_i = PI - R(S) - \sum_{i=1}^{n} (1 + \lambda_i) \cdot E[T_i(S)] - \alpha \cdot Std[R(S)].
$$

Consequently the expected profit is equal to:

$$
E[G_i] = PI - E[R(S)] - \sum_{i=1}^{n} (1 + \lambda_i) \cdot E[T_i(S)] - \alpha \cdot Std[R(S)]
$$

$$
= PI - E(S) - \sum_{i=1}^{n} \lambda_i \cdot E[T_i(S)] - \alpha \cdot Std[R(S)]
$$

We like to maximise this expected profit $E[G_i]$ while keeping the retained standard deviation $Std[R(S)]$ fixed. To optimise the profit problem we should use the Lagrange method of multipliers.

With the formal notation $w_j$ for all the unspecified parameters which one needs for describing the reinsurance protections (for example: the quota share retention, surplus line, priority for excess of loss, priority for stop loss), and $\mu^*$ for the Lagrange multiplier, we need to solve the following equations for all $j$

$$
0 = \frac{\partial E[G_i]}{\partial w_j} = \frac{\partial}{\partial w_j} \left[ PI - E[R(S)] - \sum_{i=1}^{n} (1 + \lambda_i) \cdot E[T_i(S)] + (\mu^* - \alpha) \frac{\partial}{\partial w_j} [Std[R(S)]] \right],
$$

$$
Std[R(S)] = C
$$

Setting $\mu = \mu^* - \alpha$, one sees immediately that this is the Lagrange formulation for the conditional optimisation problem defined by: $G = PI - E(S) - \sum_{i=1}^{n} \lambda_i \cdot E[T_i(S)]$ with $Var[R(S)] = C^2$. 

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Concluding, with $\Psi = E[G] + \mu \cdot \text{Var}[G]$ the Lagrange set of equations for the last formulation equals

$$\frac{\partial \Psi}{\partial w_j} = \frac{\partial}{\partial w_j} E[G] + \mu \cdot \frac{\partial}{\partial w_j} \text{Var}[G] = 0 \text{ with } \text{Var}[\mathcal{R}(S)] = C^2,$$

or equivalently,

$$- \sum_{j=1}^{\infty} \lambda_j \cdot \frac{\partial}{\partial w_j} E[T_j(S)] + \mu \cdot \frac{\partial}{\partial w_j} \text{Var}[\mathcal{R}(S)] = 0 \text{ with } \text{Var}[\mathcal{R}(S)] = C^2.$$

In the rest of the paper we will express the constraint on the variance by $\text{Std}[\mathcal{R}(S)] = x \cdot \text{Std}(S)$ with $x \in [0,1]$ (note that we assume that $\text{Std}(S) < \infty$).

This has the advantage that the possible conditional values are easier to control and the graphical representation of the optimal solutions becomes easier afterwards. The interpretation of the limited available allocated capital is also easier to apply in practice. Even if one does not know the exact allocated capital it is easier to discuss the consequences of the reinsurance program based on the retained volatility with the minimisation of the reinsurance cost, than focusing on profit margins (which are also not exactly known).

**Remark on the reinsurance loading structure in practice:**

In the (limited) range of real existing protections one has the observation that the loadings (based on realised treaties after competitive negotiations) are not highly influenced by the protection parameters. So it seems to be acceptable to work with constant loading percentages as a good first order approximation. At the other hand one observes that the loading percentages are more or less depending on: the protected branch, the covered peril (natural disaster reinsurance has other loading percentages than the basic fire risk), the country, the margins in the insurance tariff (especially for proportional reinsurance), .... This implies that, before analysing the optimisation question, one needs to know very well the reinsurance market in practice. Also the reinsurance over different branches becomes more complex.

In the next table we give an overview of the notation for the loading factors we will use afterwards. Even more important is the application of the ordering of the reinsurance protections we assume. Further, we will postulate some ranges of values that are a first estimate of some market figures. Figures that should be understood as a compromise after negotiation in a competitive reinsurance market and which are mainly related to Belgian simple risk property business. As such they are just intended as examples of possible relevant values.

<table>
<thead>
<tr>
<th>Order</th>
<th>Reinsurance protection</th>
<th>Notation</th>
<th>Working values</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 (*)</td>
<td>Quota Share 1</td>
<td>$\lambda_q$ or $\lambda_{q1}$</td>
<td>$\lambda_q \in [0.10,0.15] \sim 0.125$</td>
</tr>
<tr>
<td>2</td>
<td>Surplus</td>
<td>$\lambda_q$</td>
<td>$\lambda_q \in [0.125,0.20] \sim 0.1625$</td>
</tr>
<tr>
<td>3 (*)</td>
<td>Quota Share 2</td>
<td>$\lambda_q$ or $\lambda_{q2}$</td>
<td>See $\lambda_q$</td>
</tr>
<tr>
<td>4</td>
<td>Excess of loss per risk</td>
<td>$\lambda_x$ or $\lambda_{x1}$</td>
<td>$\lambda_x \in [0.20,0.40] \sim 0.30$</td>
</tr>
<tr>
<td>5 (*)</td>
<td>Quota Share 3</td>
<td>$\lambda_q$ or $\lambda_{q3}$</td>
<td>See $\lambda_q$</td>
</tr>
<tr>
<td>6</td>
<td>Excess of loss per event</td>
<td>$\lambda_x$ or $\lambda_{x1}$</td>
<td>$\lambda_{x1} \in [0.30,0.50] \sim 0.40$</td>
</tr>
<tr>
<td>7 (*)</td>
<td>Quota Share 4</td>
<td>$\lambda_q$ or $\lambda_{q4}$</td>
<td>See $\lambda_q$</td>
</tr>
<tr>
<td>8</td>
<td>Stop loss</td>
<td>$\lambda_y$</td>
<td>$\lambda_y \in [0.40,0.60] \sim 0.50$</td>
</tr>
<tr>
<td>9 (*)</td>
<td>Quota Share 5</td>
<td>$\lambda_y$ or $\lambda_{y5}$</td>
<td>See $\lambda_y$</td>
</tr>
</tbody>
</table>

**Table 1: Loading factors for the reinsurance protections**

In the last table we already used a natural ordering between these loading factors, which is given in next equation.

$$0 > \lambda_q > \lambda_x > \lambda_{x1} > \lambda_{x2} \geq \lambda_{x3} \geq \lambda_{x4} \geq \lambda_{x5} > 0$$

(Eq 1)
Clearly non-proportional protections are, as a percentage of the expected ceded claim, (much) more expensive than proportional protections. In general one can state that the inequalities are strict between different protections, but among the different quota share loading factors $\lambda_{q1}$ to $\lambda_{q5}$ equalities are not excluded.

Note however that in the case that no risk is retained by the cedent and the whole portfolio is ceded to only one reinsurance protection, all the loading factors should reduce to the same value (cfr line equal zero, priorities equal zero, retained proportion equal to zero).

This observation implies some dependency of the loading factors from the parameters of the protection, but as stated before, we assume that the retained optimal solutions are close to the (undefined) set of real existing protections.

3 Excess of loss after a surplus

As noted before an excess of loss will generally be applied after a surplus retention. The other way around exists but is essentially linked to facultative reinsurance, which implies an individual protection at individual conditions. So the retained risks after facultative reinsurance can, without any loss of generality, be seen as a directly underwritten risk at retained conditions.

We assume a portfolio risk described by a random variable $S$ defined by a compound process $S = \sum_{i=0}^{N} X_i$ with $N$ the random variable of the number of claims and $X_i \sim X$ for all $i$, the random variable of individual claims, which depends on an exposure value $k$ (for example sum insured) characterised by a non-negative random variable $K$ with density function $c(k)$ and a distribution function $C(k) = \int_{i=0}^{k} c(l)dl$. So we assume the individual claim $X$ is a mixture of possible claims $X(k)$ with exposure $k$.

It is important to note that $C(k)$ is the distribution function of the exposure values of the policies attacked by a claim and not the distribution function of exposure values of the insured policies. The first distribution describes the probabilities that a loss is associated with some exposure value; the second one is a static description of the relative number of insured risks with a certain exposure value in the insurance portfolio.

We suppose a surplus reinsurance with a retained line $L$ and an excess of loss priority $R$. We sometimes express the priority $R$ as a proportion $r$ of the line $L$: $r = R/L$. We denote the individual retained risk with exposure $k$ after surplus line $L$ equal to $X_L(k)$, the mixture of all the retained risks by $X_L$ and after excess of loss priority $R$ by $(X_L(k) \land R)$ and $(X_L \land R)$.

An overview of all used notations, definitions and basic derivations can be found in the annex.

The retained risk after surplus line $L$ and excess of loss priority $R$ is equal to $\mathcal{R}(L,R) = \sum_{i=0}^{N} [(X_L)_i \land R]$, with $E[\mathcal{R}(L,R)] = E(N) \cdot E(X_L \land R) = DSX_1$ and $Var[\mathcal{R}(L,R)] = E(N) \cdot \{E[(X_L \land R)^2] + A \cdot E^2(X_L \land R)\} = DSX_2$ with $A = \frac{Var(N) - E(N)}{E(N)} \in (-1, \infty)$. The total reinsurance cost is equal to $(1 + \lambda_n) \cdot E(N) \cdot [E(X) - E(X_L)] = DSX_4$ for the surplus plus the part for the excess of loss after surplus given by $(1 + \lambda_n) \cdot E(N) \cdot [E(X_L) - E(X_L \land R)] = DSX_4$. 

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If we note $\Psi(L, R) = \mu \cdot DSX_2 + \left[ -DSX_1 - DSX_3 - DSX_4 \right]$, then the Lagrange equations become

$$\frac{\partial}{\partial W} \Psi(L, R) = \mu \frac{\partial}{\partial W} DSX_2 + \frac{\partial}{\partial W} \left[ -DSX_1 - DSX_3 - DSX_4 \right] = 0 \quad \text{with} \ W = L \ or \ R.$$ 

Solving these equations leads to the following relation between $L$ and $R$ (see paragraph 7.2.1.3 Optimal solution):

$$r + A \cdot E(Y_L \wedge r) = \int_{k=L}^{\infty} c(k) E[(Y(k) \wedge r)^2] dk + A \cdot E(Y_L \wedge r) \cdot \int_{k=L}^{\infty} c(k) E[Y(k) \wedge r] dk$$

$$\int_{k=L}^{\infty} c(k) E[Y(k) \wedge r] dk - Q \cdot \int_{k=L}^{\infty} E[Y(k)] c(k) dk$$

$$= \int_{k=L}^{\infty} \left\{ E[Y(k) \wedge r] + A \cdot E(Y_L \wedge r) \right\} c(k) dk$$

With $Q = \frac{\lambda_x - \lambda_y}{\lambda_y} \in (0,1)$, $r = R/L$, $Y(k) = \frac{X(k)}{k}$ for each $k>0$ and $Y_L = \frac{X_L}{L}$ for each $L>0$. (Eq 2)

Note that for $r \to \infty$ the left hand side of (Eq 2) becomes $\infty$ and the right-hand side bounded (if $Q<1$), which implies that (for $x<1$) an excess of loss protection will always be used (except for $L=0$, implying that $x$ should also be zero).

However until now we did not use the variance constraint $Var[\mathcal{R}(L, R)] = x^2 Var(S)$, $x \in [0,1]$ or equivalently:

$$E[(X_L \wedge R)^2] + A \cdot E^2(X_L \wedge R) = x^2 \cdot [E(X^2) + A \cdot E^2(X)] \quad \text{with} \ x \in [0,1], \text{ or equivalent}$$

$$E[(X_L \wedge R)^2] + A \cdot E^2(X_L \wedge L \cdot r) = x^2 \cdot [E(X^2) + A \cdot E^2(X)] \quad \text{and (Eq 3)}$$

For a given line $L$ this formula has a unique solution for $R$, if in case $R = \infty$ the left hand side of the formula is greater than the right-hand side. Analogously one can prove that under a weak condition the formula has a unique solution for $L$, given $R$, if for $L = \infty$ the left-hand side is greater than the right hand side. See equations (Eq 30), (Eq 31) and (Eq 32).

To find a feasible solution for $L$ and $R$ we used an iterative process:

- Step 0: Choose a starting value $L$ (for example by solving (Eq 3) for $L$ with $R = L$ or $R = \infty$).
- Step 1: For the resulting line $L$ solve (Eq 2) for $r$.
- Step 2: For the resulting priority $r$ solve (Eq 3) for $L$. If $L$ becomes $\infty$ then solve (Eq 3) for $R$ and stop the iteration.
- Step 3: Repeat until convergence in $R$ and $L$.

It is not clear if these equations will generate an unique solution, or if local optima are possible. Probably one needs to add some conditions on the behaviour of $X(k)$ or $Y(k)$ in relation to $K$.

### 3.1 Special case $A=0$

Note that the Poisson process belongs to this case for which (Eq 2) reduces to
Remark that this equation in $r$ is independent on the claim number process, but stays depending on the line $L$ and the exposure distribution $K$.

Note also that $r > 0$ implies that $\int_{k=L}^{\infty} \{E[Y(k) \wedge r] - Q \cdot E[Y(k)]\} \cdot c(k) dk > 0$ should hold.

The variance constraint here reduces to $E[(X_L \wedge L \cdot r)^2] = x^2 \cdot E(X^2)$ with $x \in [0,1]$.

### 3.2 Special case $X(k) = k \cdot Z$

This implies that $Y(k) = \frac{X(k)}{k} \sim Z$ for all $k$ and (Eq 2) reduces to (see (Eq 33) & (Eq 34))

$$r + A \cdot E(Y_L \wedge r) = \frac{E[\{Z \wedge r\}^2] + A \cdot E(Y_L \wedge r) \cdot E[Z \wedge r]}{E[Z \wedge r] - Q \cdot E[Z]}$$

or equivalently

$$r = \frac{E[\{Z \wedge r\}^2] + A \cdot E(Y_L \wedge r) \cdot Q \cdot E[Z]}{E[Z \wedge r] - Q \cdot E[Z]}$$

Remark that this equation in $r$ stays dependent on the claim number process through the factor $A$, but also of the line $L$ and the exposure distribution $K$ (both through the factor $E(Y_L \wedge r)$).

Note that if $E(Z \wedge r) - Q \cdot E(Z) > 0$ then $r > 0$. This condition is true if we assume that $r \in (0, \infty)$ because $r + A \cdot E(Y_L \wedge r)$ is always positive (for $A \geq 0$ is this obvious, but is also true for $A \in [-1,0]$ because $E(Y_L \wedge r) \leq E(Z \wedge r) \leq r$ (see (Eq 33)), which implies that the left hand side of (Eq 5) and also the numerator of the right hand side of (Eq 5) will be positive. We can conclude that the denominator should always be positive.

The variance constraint further reduced, but is not fundamentally changed to

$E[(X_L \wedge R)^2] + A \cdot E^2(X_L \wedge R) = x^2 \cdot [E(Z^2) \cdot E(K^2) + A \cdot E^2(Z) \cdot E^2(K)]$

### 3.3 Special case $A=0$ and $X(k) = k \cdot Z$

$$r = \frac{E[\{Z \wedge r\}^2]}{E[Z \wedge r] - Q \cdot E[Z]}$$

Remark that this equation in $r$ is independent on the line $L$, independent on the claim number process and also independent on the exposure distribution $K$.

Once the priority $r$ is known one can solve for $L$ with the help of the variance constraint. This constraint reduces further a little, but the most important advantage is that both equations separately of each other can be solved. The variance constraint becomes: $E[(X_L \wedge R)^2] = x^2 \cdot E(Z^2) \cdot E(K^2)$ for $x \in [0,1]$. 
3.4 QS as a special case of a Surplus

If we define an exposure measure \( k \) equal to 1 for all the risks in the insured portfolio and the line \( L \) a value belonging to the interval \([0,1]\), then the corresponding surplus reinsurance is a quota share with \( L \) the retained proportion for all risks.

For this exposure measure we become that: \( Y(k)=X(k)\sim X \) for all \( k \), \( X_L = L \cdot X \), \( Y_L \sim X \), \( r = R \) and \( c(k) = 0 \) for all \( k \) except for \( k = 1 \) \( c(1) = 1 \). This reduces (Eq 2) to

\[
R = -A \cdot E(X \land R) + \frac{E[(X \land R)^2] + A \cdot E^2(X \land R)}{E(X \land R) - Q \cdot E(X)} \quad \text{with} \quad 0 < Q = \frac{\lambda_x - \lambda_y}{\lambda_x} < 1 \quad \text{(Eq 8)}
\]

The variance constraint \( Var[R(L, R)] = x^2 Var(S) \), \( x \in [0,1] \) becomes:

\[
L^2 \cdot E[(X \land R)^2] + A \cdot E^2(X \land R) = x^2 \cdot [E(X^2) + A \cdot E^2(X)] \quad \text{with} \quad x \in [0,1] \text{ and } L \in [x,1] \quad \text{(Eq 9)}
\]

3.5 Optimal solution: graphical presentation

To illustrate the optimal solution for a surplus-excess of loss combination in practice, we will use an example of the special case \( X(k)=k \cdot Z \) with \( A \) different from 0.

In Table 2 we give an overview of the most important kernel-figures, which will characterise our claim statistic. These numbers and distribution fits are based on a practical case-study in fire insurance (handled by Aon Re Belgium).

We will use two lognormal distributions to describe the exposure value \( K \) (= sum insured values of the claims) and the claim severity per unit of exposure value \( Z \). Afterwards we will compare some of the results with a double Burr model to illustrate the impact of model choice.

<table>
<thead>
<tr>
<th></th>
<th>N</th>
<th>K</th>
<th>Z</th>
<th>X</th>
<th>S</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>1,225.000</td>
<td>125.000</td>
<td>0.0180</td>
<td>2.250</td>
<td>2,756.250</td>
</tr>
<tr>
<td>Std</td>
<td>52.500</td>
<td>200.000</td>
<td>0.0900</td>
<td>21.530</td>
<td>762.738</td>
</tr>
<tr>
<td>Cv</td>
<td>0.04286</td>
<td>1.60000</td>
<td>5.00000</td>
<td>9.56870</td>
<td>0.27673</td>
</tr>
<tr>
<td>Skewness</td>
<td>7.2</td>
<td>7.2</td>
<td>280</td>
<td>280</td>
<td>280</td>
</tr>
<tr>
<td>Burr</td>
<td>8.896</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Lognormal</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Value A</td>
<td>1.25</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Value Q</td>
<td>(0.30-0.1625)/0.30 = 11/24 = 0.45833</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 2: Description of the claim statistic

In Figure 1, we bring together the most important results. As a function of the retained proportion of standard deviation \( x \), one has the optimal line (= \( L_i \)) and the optimal priority (= \( R_i \)). To compare the optimal priority with an excess of loss protection without surplus reinsurance the corresponding priority \( R_0 \) is also plotted. These values are plotted on a logarithmic scale on the left axes. The plot illustrates clearly that with a low-level of reinsurance both priorities are nearly the same, indicating that the surplus protection has nearly no influence on the retention.

On the right axes the ratio is (= \( M_i = r \)) between the priority \( R_i \) and the line \( L_i \) plotted and compared with the constant solution corresponding with \( A \) equal to zero (=\( M_0A \)). This ratio increases for decreasing proportion of retained risk, implying that the surplus reinsurance becomes more important for high-level reinsurance.
To illustrate the relative importance of the two protections, the reinsurance loadings per protection ($\lambda_p, \lambda_s$) and the expected claim transfer (equal $E[N] \cdot (E[X] - E[X_L])$ and $E[N] \cdot (E[X_L] - E[X_L \wedge R])$) are plotted as a function of the retained proportion of the standard deviation (see Figure 2). Here it becomes clear that for a high-level of reinsurance one will use less excess of loss protection in favour of the surplus protection.

If we compare the results with two different models (which has the same first 2 moments, but a different skewness), then one becomes different optimal solutions.

In this example one can remark that:

- The proportion between priority and line ($= r = M_i$) is not so different, about 5%.
- The value of the line ($= L_i$) is relatively strongly influenced, the priority less.
- The reinsurance loading increases especially for low-level reinsurance protections.
- Also the relative subdivision between the two protections has strongly changed.

### Table 1: Double Lognormal model vs. Double Burr model vs. Burr / Lognormal model

<table>
<thead>
<tr>
<th>$x$</th>
<th>0.1</th>
<th>0.3</th>
<th>0.5</th>
<th>0.7</th>
<th>0.1</th>
<th>0.3</th>
<th>0.5</th>
<th>0.7</th>
<th>0.1</th>
<th>0.3</th>
<th>0.5</th>
<th>0.7</th>
</tr>
</thead>
<tbody>
<tr>
<td>MOA</td>
<td>0.18998</td>
<td>0.18998</td>
<td>0.18998</td>
<td>0.18998</td>
<td>0.24333</td>
<td>0.24333</td>
<td>0.24333</td>
<td>0.24333</td>
<td>1.281</td>
<td>1.281</td>
<td>1.281</td>
<td>1.281</td>
</tr>
<tr>
<td>$M_i$</td>
<td>0.22031</td>
<td>0.20244</td>
<td>0.19410</td>
<td>0.19106</td>
<td>0.27068</td>
<td>0.25721</td>
<td>0.24908</td>
<td>0.24508</td>
<td>1.229</td>
<td>1.271</td>
<td>1.283</td>
<td>1.283</td>
</tr>
<tr>
<td>$L_i$</td>
<td>65.643</td>
<td>381.506</td>
<td>1,450.970</td>
<td>5,897.050</td>
<td>60.775</td>
<td>296.046</td>
<td>941.784</td>
<td>3,384.424</td>
<td>0.926</td>
<td>0.776</td>
<td>0.649</td>
<td>0.574</td>
</tr>
<tr>
<td>$R_i$</td>
<td>14,462</td>
<td>77,231</td>
<td>281,638</td>
<td>1,126,705</td>
<td>16,451</td>
<td>76,145</td>
<td>234,584</td>
<td>829,467</td>
<td>1.137</td>
<td>0.986</td>
<td>0.833</td>
<td>0.736</td>
</tr>
</tbody>
</table>
This example shows that, in the context of optimal reinsurance, the model choice is quite an important issue. Of course, also the estimates of the parameters (model and reinsurance loading structure) will influence the optimised results.

4 Excess of loss and quota share

Here the retained risk equals to $R(a, R) = \sum_{i=0}^{N} (a \cdot X_{i} \wedge a \cdot R) = a \sum_{i=0}^{N} (X_{i} \wedge R)$, if we express the excess of loss priority $R$ in terms of the 100% claim size.

The left-hand side retained risk formula expresses that we firstly apply the quota share and secondly the excess of loss with priority $R$. The right-hand side formula expresses that we firstly apply the excess of loss priority $R$ and secondly the quota share with retention $a$.

It is clear that both applications will give the same retained risk for the cedent, but nevertheless the difference in application order is fundamentally different. The first application (corresponding to the left-hand side) says that the quota share reinsurers are not participating in the excess of loss protection. Not participating means: not participating in the advantages (= recuperating the excess of loss intervention) and not participating in the disadvantages (= participating in the excess of loss premium).

The second application (corresponding to right-hand side) says that the quota share reinsurers also participate in the excess of loss protection. This should imply that they participate as proportional reinsurer in the excess of loss premium (implied by the fortune’s sharing principle of the quota share reinsurance), which in practice is nearly always the case.

A remark concerning the impact of the fortune’s sharing principle is necessary, because it generates a serious contradiction with one of our basis assumptions. If the quota share reinsurer pays his part in the common excess of loss protection, then this will increase its cost structure. To compensate this additional cost he needs to increase the quota share-loading factor. To be sure that he will obtain the same result, this increase should (nominally) be equal to the excess of loss loading (in theory this loading could be smaller due to less management costs and lower volatility, but we will neglect this potential marginal reduction). But both loadings are (in the framework of this paper) not on the same basis, which implies that for different excess of loss priorities one becomes different loading factors. This observation means that the "constant mean value premium principle" hypothesis is not valid any more.

Nevertheless, this problem can be avoided in an elegant way by assuming that the excess of loss premium is
paid directly by the cedent. This instead of passing the excess of loss premium through the quota share contract and recuperate it in an indirect way through the loading structure. This interpretation exists (very rarely) in practice but the reverse is the standard. However the alternative interpretation gives (in the framework of this paper) the same (theoretical) result as the logical loading correction gives, nearly the same result as in practice and coincides perfectly with the "constant mean value premium principle".

4.1 Excess of loss after quota share (QS-XL)

As stated before the retained risk is equal to \( \mathcal{R}(a, R) = \sum_{i=0}^{N} (a \cdot X_i \land a \cdot R) = a \sum_{i=0}^{N} (X_i \land R) \), with

\[
E[\mathcal{R}(a, R)] = a \cdot E(N) \cdot E(X \land R) = DQX_1 \quad \text{and with} \quad A = \frac{[Var(N) - E(N)]}{E(N)} \in (-1, \infty).
\]

\[
Var[\mathcal{R}(a, R)] = a^2 \cdot [E(N) \cdot E((X \land R)^2)] + A \cdot E(N) \cdot E^2(X \land R) = DQX_2
\]

The total reinsurance cost is equal to \( (1 - a) \cdot (1 + \lambda_x) \cdot E(N) \cdot E(X) = DQS \), for the quota share plus the part for the excess of loss after quota share given by \( (1 + \lambda_x) \cdot E(N) \cdot [E(a \cdot X) - E(a \cdot X \land a \cdot R)] = QDX_4 \).

From the preceding section it follows as a special case from the surplus formulae (see (Eq 8)) and by interpreting the numerator as the so-called normalised portfolio variance (see (Eq 25)) that

\[
R + A \cdot E(X \land R) = \frac{Vrn(X \land R)}{E(X \land R) - Q \cdot E(X)} \quad \text{with} \quad 0 < Q = \frac{\lambda_x - \lambda_q}{\lambda_q} < 1
\]

(Eq 10)

This equation (Eq 10) is quite interesting to compare with (Eq 2), (Eq 5) and (Eq 7). The nominator of the right part of equation (Eq 2) could be understood as some generalised weighted variance formula. Equation (Eq 5) is very well comparable with (Eq 10), the only difference is generated by the factor \( E(Y_i \land r) \) and the fact that the role of \( X \) and \( R \) is replaced by \( Z \) and \( r \). Anyway, it is remarkable how close these two total different reinsurance programs come together (see also special case (Eq 7)).

We can conclude that the optimal excess of loss priority is independent on the quota share protection and in first instance also independent on the variance constraint (if \( R \) implies that \( a \leq 1 \), see further).

Due to the fact that \( Vrn(X \land R) \geq 0 \) and \( R \geq A \cdot E(X \land R) \) for \( A \in (-1, \infty) \) one has that a feasible solution \( R \) should fulfil the inequality \( E(X \land R) - Q \cdot E(X) \geq 0 \) implying that \( R \geq Q \cdot E(X) \).

Note that for \( R \rightarrow \infty \) the left-hand side of the equation becomes \( \infty \) and the right hand side limited (if \( Q < 1 \)), which implies that (for \( x < 1 \)) an excess of loss protection will always be used.

The variance constraint \( Var[\mathcal{R}(a, R)] = x^2 \cdot Var(S) \), \( x \in [0,1] \) becomes:

\[
a^2 \cdot [E((X \land R)^2)] + A \cdot E^2(X \land R) = x^2 \cdot [E(X^2) + A \cdot E^2(X)] \quad \text{with} \quad x \in [0,1],
\]

implying \( a \in [x,1] \).

(Eq 11)

Note that for \( a = 1 \) (= no quota share protection) only the next equation has to be solved for \( R \):

\[
[E((X \land R)^2)] + A \cdot E^2(X \land R) = x^2 \cdot [E(X^2) + A \cdot E^2(X)] \quad \text{or}
\]

\[
Vrn(X \land R) = x^2 \cdot Vrn(X)
\]

(Eq 12)

To find a feasible solution for \( a \) and \( R \) one can:
• Firstly solve (Eq 10) for $R$.
• Secondly check with (Eq 11) if the corresponding $a$ value is less than or equal to 1. If the answer is positive one will retain the $R$(Eq 10) and $a$(Eq 10 & (Eq 11)) as the optimal combined solution.
• Thirdly, if the check is negative one needs to solve (Eq 12) for $R$ and retain $R$(Eq 12) together with $a=1$ as the optimal combined solution.

4.2 Quota share after Excess of loss (XL-QS)

We illustrate for this combination the "constant mean value principle problem" explicitly. So, if the quota share reinsurer increases his loading with the corresponding proportion in the excess of loss loading one obtains the following reinsurance cost:

- For the excess of loss protection $(1 + \lambda_q) \cdot E(N) \cdot [E(X) - E(X \wedge R)]$
- For the quota share protection $(1 - a) \cdot (1 + \lambda_q) \cdot E(N) \cdot E(X \wedge R)$
- From the excess of loss loading, the quota share reinsurers will directly recuperate the corresponding proportional part $(1 - a) \cdot \lambda_q \cdot E(N) \cdot [E(X) - E(X \wedge R)]$, but we assume that they will recuperate this additional cost indirectly through the loading structure.

The retained risk is equal to $R(a, R) = a \sum_{i=0}^{N} (X_i \wedge R)$, which implies again the same formulas $DQX_1 = DXQ_1$ and $DQX_2 = DXQ_2$ as stated in section 4.1.

The reinsurance cost is equal to $(1 + \lambda_q) \cdot E(N) \cdot [E(X) - E(X \wedge R)] = DXQ_1$ for the excess of loss (before quota share) plus for the quota share $(1 - a) \cdot (1 + \lambda_q) \cdot E(N) \cdot E(X \wedge R) = DXQ_2$.

Instead of solving the Lagrange equations, we will derive the solution of the problem (just to illustrate a fully equivalent way of working) by setting the derivative of the expected profit with respect to $a$ equal to zero while keeping the variance constraint under control through the parameter $R$.

Note also that the profit function $G(a,R)$ is defined as: $G(a, R) = PI - R(S) - \sum_{i=1}^{n} (1 + \lambda_q) \cdot E[T_i(S)]$
and $E[G(a, R)] = PI - E(S) - E(N) \cdot \{ \lambda_q \cdot (1 - a) \cdot E(X \wedge R) + \lambda_q \cdot [E(X) - E(X \wedge R)] \}$

From the variance equation $\lambda_q \cdot (1 - a) \cdot E(X^2) + (1 - a) \cdot (1 + \lambda_q) \cdot E(N) \cdot E(X \wedge R) = \lambda_q \cdot (1 - a) \cdot [E(X^2) + A \cdot E^2(X)]$, we derive that
(with $R'(a) = \frac{\partial}{\partial a} R(a)$):

$$R'(a) = -\frac{x}{a} \frac{Vrn(X)}{[R(a) + A \cdot E(X \wedge R(a))] \cdot \overline{F}(R(a))} = -\frac{1}{a} \frac{Vrn(X \wedge R(a))}{[R(a) + A \cdot E(X \wedge R(a))] \cdot \overline{F}(R(a))} \tag{Eq 13}$$

$$\frac{\partial E[G(a, R)]}{\partial a} = E(N) \cdot \{-\lambda_q \cdot (1 - a) \cdot \overline{F}(R) \cdot R'(a) + \lambda_q \cdot E(X \wedge R) + \lambda_q \cdot \overline{F}(R) \cdot R'(a) \}$$

$$= \lambda_q \cdot E(N) \cdot \left\{ E(X \wedge R) - (1 + \frac{Q}{a}) \cdot \frac{Vrn(X \wedge R)}{[R + A \cdot E(X \wedge R)]} \right\}.$$ 

This equation is equal to zero if
$$a = Q \cdot \frac{E(X \wedge R)^2 + A \cdot E^2(X \wedge R)}{R \cdot E(X \wedge R) - E(X \wedge R)^2} \text{ with } Q = \frac{\lambda_q - \lambda_q}{\lambda_q} \in (0,\infty) \tag{Eq 14}$$
Note that for $R \to \infty$ the right-hand side of the equation becomes 0 and the left-hand side stays limited to $a \in [0, \infty)$ (if $Q>0$), which implies that (for $x<1$) an excess of loss protection will always be used (except for $a=0$, implying that $x$ should also be 0).

If we combine last equation with the variance constraint (Eq 11), one finds

$$Q^2 \frac{E[(X \land R)^2] + A \cdot E^2 (X \land R)}{R \cdot E(X \land R) - E[(X \land R)^2]} = x^2 \left[ E(X^2) + A \cdot E^2(X) \right].$$

Due to $R \cdot E(X \land R) - E[(X \land R)^2] \geq 0$ and also the other terms are positive (see variance formula for a compound process) we can conclude that:

$$R \cdot E(X \land R) - E[(X \land R)^2] = Q \sqrt{\frac{E[(X \land R)^2] + A \cdot E^2 (X \land R)}{E(X^2) + A \cdot E^2(X)}} \Rightarrow$$

$$R = \frac{E[(X \land R)^2]}{E(X \land R)} + \frac{Q}{x \cdot E(X \land R)} \sqrt{\frac{E[(X \land R)^2] + A \cdot E^2 (X \land R)}{E(X^2) + A \cdot E^2(X)}}$$

$$= -A \cdot E(X \land R) + \frac{Vrn(X \land R)}{E(X \land R)} \left[ 1 + \frac{Q \cdot Vrn(X \land R)}{x \cdot Vrn(X)} \right], \quad \text{with } Q = \frac{\lambda_x - \lambda_{eq}}{\lambda_{eq}}, \quad (0, \infty)$$

Note that this equation depends only on the unknown $R$ and is independent on the unknown $a$.

To find a feasible solution for $a$ and $R$ one can:

- Firstly solve (Eq 15) for $R$.
- Secondly check with (Eq 11) if the corresponding $a$ value is less than or equal to 1. If the answer is positive one will retain the $R$ (Eq 15) and $a$ (Eq 11) & (Eq 15) as the optimal combined solution.
- Thirdly, if the check is negative one needs to solve (Eq 12) for $R$ and retain $R$ (Eq 12) together with $a=1$ as the optimal combined solution.

### 4.3 Stop loss as a special case of Excess of loss

If the number of claims is fixed (which corresponds with $A = -1$ and $N = n$), then the excess of loss protection is reduced to a fixed sum of stop loss protections. For $n=1$ the excess of loss reduces to a stop loss.

As stated before the retained risk is equal to $\mathcal{R}(a, R) = \sum_{i=0}^{n} (a \cdot X_i \land a \cdot R) = a \sum_{i=0}^{n} (X_i \land R)$, with $E[\mathcal{R}(a, R)] = a \cdot n \cdot E(X \land R)$ and $\text{Var}[\mathcal{R}(a, R)] = a^2 \cdot \left[n \cdot E[(X \land R)^2] - n \cdot E^2(X \land R)\right]$. It is clear that both equations are a n-multiple of a stop loss formulation.

#### 4.3.1 Stop loss after quota share (QS-SL)

Equation (Eq 10) reduces now to

$$P = E(S \land P) + \frac{\text{Var}(S \land P)}{E(S \land P) - Q \cdot E(S)} \quad \text{with} \quad 0 \leq Q = \frac{\lambda_x - \lambda_{eq}}{\lambda_x} \leq 1$$

(Eq 16)

This implies that the optimal stop loss priority is independent on the quota share protection and in first instance also independent on the variance constraint (if $P$ implies that $a \leq 1$, see further).

Note that for $P \to \infty$ the left-hand side of the equation becomes $\infty$ and the right-hand side limited (if $Q<1$).
which implies that (for \( x < 1 \)) a stop loss protection will always be used.

The variance constraint (Eq 11) becomes
\[
a = x \frac{\text{Std}(S)}{\text{Std}(S \land P)} \quad \text{with} \quad x \in [0.1] \quad \text{but also} \quad a \in [x,1]. \tag{Eq 17}
\]

### 4.3.2 Quota share after stop loss (SL-QS)

Equation (Eq 14) reduces to
\[
a = Q \frac{\text{Var}(S \land P)}{P \cdot E(S \land P) - E[(S \land P)^2]} \quad \text{with} \quad Q = \frac{\lambda_x - \lambda_q}{\lambda_q} \quad (0 \leq Q < \infty). \tag{Eq 18}
\]

From \( E[(S \land P)^2] \leq P \cdot E(S \land P) \) it follows that \( a \geq 0 \).

Note that for \( P \to \infty \) the right-hand side of the equation becomes zero and the left-hand side stays limited to \( a \in [0, \infty) \) (if \( Q > 0 \)), which implies that (for \( x < 1 \)) a stop loss protection will always be used (except for \( a = 0 \), implying that \( x \) should also be zero).

The equation (Eq 15) becomes
\[
P = \frac{E[(S \land P)^2]}{E(S \land P)} + \frac{1}{E(S \land P)} \cdot x \sqrt{\frac{\text{Var}(S \land P)}{\text{Var}(S)}} \quad \text{or} \quad P = E(S \land P) + \frac{\text{Var}(S \land P)}{E(S \land P)} \left[ 1 + \frac{Q}{x \sqrt{\text{Var}(S)}} \right], \quad \text{with} \quad 0 \leq Q = \frac{\lambda_x - \lambda_q}{\lambda_q} < \infty \tag{Eq 19}
\]

Note that this equation depends only on the unknown \( P \) and is independent on the unknown \( a \).

## 5 Surplus and quota share

Combining quota share and surplus protections is a combination of two proportional reinsurance protections. This implies that the remarks concerning the “constant mean value premium principle” hypothesis are not relevant for this combination.

For the description of this combination one can choose between the “individual model” and the “collective model”. In the context of this paper, we will limit us to the “collective model”.

Due to
\[
a \cdot X_L(k) = \begin{cases} \frac{a \cdot X(k)}{k} & \text{if} \quad k \leq L \\ \frac{L}{a} X(k) & \text{if} \quad k > L \end{cases}
\]

we can conclude that \( a \cdot X_L(k) = (a \cdot X)_{(a \cdot L)}(k) \) and consequently \( a \cdot X_L = (a \cdot X)_{(a \cdot L)} \).

The last expression explains that the order of application for a surplus quota share combination has no influence on the retained risk.

We suppose a surplus reinsurance with a retained line \( L \) (expressed in terms of 100% claim size) and a quota share with retention \( a \).

We denote the individual retained risk with exposure \( k \) after surplus line \( L \) equal to \( X_L(k) \) and the mixture
of all the retained risks by $X_L$. The retained portfolio risk after surplus line $L$ and quota share with retention $a$ is, in the collective model, equal to $\mathcal{R}(a, L) = a \sum_{i=0}^{N} (X_{i,L}) = \sum_{i=0}^{N} (a \cdot X_{i}) = \sum_{i=0}^{N} \left[ (a \cdot X)_{(a,L)} \right]$.

5.1 Surplus after quota share (QS-SPL)

$$E[\mathcal{R}(a, L)] = a \cdot E(N) \cdot E(X_{L}) = DQS_1$$ and with $A = \frac{[Var(N) - E(N)]}{E(N)} \in (-1, \infty)$.

$$Var[\mathcal{R}(a, L)] = a^2 \cdot E(N) \cdot \left[ E\left[ X_{L}^2 \right] + A \cdot E^2(X_{L}) \right] = DQS_2$$.

The total reinsurance cost is equal to $(1 - a) \cdot (1 + \lambda_q) \cdot E(N) \cdot E(X) = DQS_3$ for the quota share, plus for the part for the surplus after quota share $(1 + \lambda_q) \cdot a \cdot E(N) \cdot \left[ E(X) - E(X_{L}) \right] = DQS_4$.

If we note $\Psi(a, L) = \mu \cdot DQS_2 + [-DQS_1 - DQS_3 - DQS_4]$, then the Lagrange equations become:

$$\frac{\partial}{\partial W} \Psi(a, L) = \mu \frac{\partial}{\partial W} DQS_2 + \frac{\partial}{\partial W} [-DQS_1 - DQS_3 - DQS_4] = 0 \quad \text{with} \quad W = a \text{ or } L$$.

Which implies

$$L = \left[ -A \cdot E(X_{L}) + \frac{E\left[ X_{L}^2 \right]}{E(X_{L}) - Q \cdot E(X)} \right] \int_{k=L}^{\infty} \frac{E[Y(k)] \cdot c(k) dk}{E[Y^2(k)] \cdot c(k) dk}$$

(Eq 20)

with for all $k$ $Y(k) = \frac{X(k)}{k}$ and $Q = \frac{\lambda_q - \lambda_q}{\lambda_q} \in (0, 1)$

This implies that the optimal surplus priority is independent on the quota share protection and in first instance also independent on the variance constraint (if $L$ implies that $a \leq 1$, see further).

Remark that last equation has a strong similarity with the excess of loss case (Eq 10 (the role of the priority $R$ is largely taken over by the line $L$) if we interpret the numerator as the so-called normalised portfolio variance after surplus protection. The most important difference is induced by the ratio factor in the right-hand side.

The variance constraint $Var[\mathcal{R}(a, L)] = x^2 Var(S), \quad x \in [0,1]$ becomes:

$$a^2 \cdot \left[ E\left[ X_{L}^2 \right] + A \cdot E^2(X_{L}) \right] = x^2 \cdot \left[ E(X^2) + A \cdot E^2(X) \right]$$ with $x \in [0,1]$ and $a \in [0,1]$.

or $a^2 \cdot Var(X_{L}) = x^2 \cdot Var(X)$

(Eq 21)

Note that for $a=1$ (= no quota share protection) only the next equation has to be solved for $L$:

$$E\left[ X_{L}^2 \right] + A \cdot E^2(X_{L}) = x^2 \cdot \left[ E(X^2) + A \cdot E^2(X) \right] \quad \text{or} \quad Var(X_{L}) = Var(X)$$

(Eq 22)

The variance of the retained portfolio after a surplus protection is in general not a non-decreasing function for the line $L$, implying that (Eq 22) not always possesses a unique solution. However we will further suppose that for $A \in [-1,0]$ the (weak) condition (Eq 43) of positive derivative is always fulfilled. This condition implies that (Eq 21) and (Eq 22) has a unique solution for $L$ and from (Eq 20) follows that the set of feasible solutions for $L$ fulfil the condition that $E(X_{L}) > Q \cdot E(X)$ implying that $Q \cdot \frac{E(X)}{E(Y)} < L$ (see (Eq 26)).

To find a feasible solution for $a$ and $L$ one can:

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First solve Eq 20 for \( L \) within the set of \( \{ L > L' ; L' \text{ solution of } E(X_L) - Q \cdot E(X) = 0 \} \).

Secondly check with Eq 21 if the corresponding \( a \) value is less than or equal to 1. If the answer is positive one will retain the \( L \) (Eq 20) and \( a \) (Eq 20) & (Eq 21) as the optimal combined solution.

Third, if the check is negative one needs to solve Eq 22 for \( L \) and retain “L(Eq 22)” together with \( a=1 \) as the optimal combined solution.

### 5.2 Quota share after Surplus (SPL-QS)

The retained risk is equal to

\[
R(a, L) = a \sum_{i=0}^{N} (X_L)_i = \sum_{i=0}^{N} (a \cdot X_L)_i = \sum_{i=0}^{N} [(a \cdot X)_{(a,L)}]_i
\]

which implies the same formulas \( DQS_1 = DSQ_1 \) and \( DQS_2 = DSQ_2 \) as stated in section 5.1.

The reinsurance cost is equal to

\[
(1 + \lambda) \cdot E(N) \cdot [E(X) - E(X_L)] = DSQ_3
\]

for the surplus (before quota share) plus for the quota share \((1 - a) \cdot (1 + \lambda) \cdot E(N) \cdot E(X_L) = DSQ_4\).

If we note \( \Psi(a, L) = \mu \cdot DSQ_2 + [-DSQ_1 - DSQ_3 - DSQ_4] \), then the Lagrangian equations become:

\[
\frac{\partial}{\partial W} \Psi(a, L) = \mu \frac{\partial}{\partial W} DSQ_2 + \frac{\partial}{\partial W} [-DSQ_1 - DSQ_3 - DSQ_4] = 0 \quad \text{with } W = a \text{ or } L.
\]

Which implies with \( Q = \frac{\lambda - \lambda_n}{\lambda_q} \in (0, \infty) \)

\[
a = Q \left[ E(X_L^2) + A \cdot E^2(X_L) \right] \left[ \int_{L=k_L}^{\infty} E[Y^2(k)] \cdot c(k)dk \right]^{-1} = Q \left\{ \int_{L=k_L}^{\infty} E[Y(k) \cdot c(k)] \cdot \left( E(X_L) - E(X_L^2) \right) \right\}^{-1} \quad \text{(Eq 23)}
\]

The quota share retention \( a \) can never be negative, which implies that the line \( L \) should satisfy that

\[
\int_{L=k_L}^{\infty} E[Y^2(k)] \cdot c(k)dk
\]

\[
\frac{\int_{L=k_L}^{\infty} E[Y(k) \cdot c(k)] \cdot \left( E(X_L) - E(X_L^2) \right)}{\int_{L=k_L}^{\infty} E[Y(k) \cdot c(k)]} > 0.
\]

This inequality is stronger than the (weak) condition (Eq 43) used in preceding paragraph, because \( E(X_L^2) \geq E^2(X_L) \). From now on we will assume that this (stronger) condition is fulfilled.

The variance constraint \( Var[R(a, L)] = x^2 Var(S), \quad x \in [0,1] \) is equivalent with equation (Eq 21) Combining equation (Eq 21) and (Eq 23) and using the Vrn notation one becomes that

\[
\left[ \int_{L=k_L}^{\infty} E[Y^2(k)] \cdot c(k)dk \right]^2 - \left( \int_{L=k_L}^{\infty} E[Y(k) \cdot c(k)] \cdot \left( E(X_L) - E(X_L^2) \right) \right)^2 = Q^2 \cdot \frac{Vrn^3(X_L)}{Vrn^2(X)}
\]

\[
\frac{\int_{L=k_L}^{\infty} E[Y(k) \cdot c(k)]} {\int_{L=k_L}^{\infty} E[Y(k) \cdot c(k)]} 
\]

The expression between the brackets at left-hand side is by assumption positive, which implies
\[
\int_{k=L}^{\infty} E[Y^2(k)] \cdot c(k)dk
\]
\[
\int_{k=L}^{\infty} E[Y(k)] \cdot c(k)dk
\]
\[
L_{k=L} = -A \cdot E(X_L) + \frac{Vrn(X_L)}{E(X_L)} \cdot \left[ 1 + \frac{Q}{x} \cdot \sqrt{\frac{Vrn(X_L)}{Vrn(X)}} \right], Q = \frac{\lambda_x - \lambda_y}{\lambda_y} \in (0, \infty) \quad \text{(Eq 24)}
\]

This equation is fully comparable with the excess of loss case, see (Eq 15). The role of the priority \( R \) is largely taken over by the line \( L \), if we interpret the nominator as the so-called normalised portfolio variance after surplus protection. The most important difference is induced by the ratio factor in the left-hand side.

To find a feasible solution for \( a \) and \( L \) one can:
- First solve (Eq 24) for \( L \).
- Secondly check with (Eq 23) if the corresponding \( a \) value is less than or equal to 1. If the answer is positive one will retain the \( L \) (Eq 24) and \( a \) (Eq 23 & Eq 24) as the optimal combined solution.
- Third, if the check is negative one needs to solve (Eq 22) for \( L \) and retain \( L \) (Eq 22) together with \( a = 1 \) as the optimal combined solution.

### 6 Conclusions and general remarks

- For all the reinsurance combinations described in this paper, the optimal solutions depend on the loading factors through a factor \( Q \), which is of the form of \( Q(\lambda_u, \lambda_p) = \frac{\lambda_u - \lambda_p}{\lambda_p} \) or \( \frac{\lambda_u - \lambda_p}{\lambda_u} \). This implies that the optimal solutions depend only on the relative loading structure, because \( Q(\lambda \cdot \lambda_u, \lambda \cdot \lambda_p) = Q(\lambda_u, \lambda_p) \) for all \( \lambda > 0 \).

This is an advantage in practice, because it is much easier to obtain an estimate of the relative loading structure then of the absolute structure.

- In this paper we limit us to combinations of two reinsurance protections. A longer chain of reinsurance protections should receive more attention in the future. Also the optimal combination of an excess of loss with a stop loss protection is still an open problem.

- We derived only the results for the protection of one portfolio. However, the generalisation to more independent portfolios is straightforward (see for instance Bühlmann (1970) and Schmitter (2001)).

- From this paper is it clear that the order of the reinsurance protection chain is important if one uses a constant loading proportional with the expected risk ceded to the reinsurers. This order dependency makes the question of optimal reinsurance more complex. Also the heterogeneity of the portfolio has an important impact on the optimisation question. At least it will make the description and the computation much more complex. At the other hand it gives the opportunity to analyse a surplus treaty more efficiently.

- If we look to the combinations containing a quota share, then for a low level of reinsurance protection no quota share is needed. However, for the surplus - excess of loss combination both protections exist always together, but the impact of the surplus is negligible for low levels of reinsurance protection and even zero if the exposure values \( K \) are limited (which is always the case in practice).
7 Annex

7.1 Definitions, notations and well known results

• A random variable $X$ assumes certain real numerical values in accordance with well-defined probabilities. We speak of the law of probability for a given random variable. This law can be described by means of the (cumulative) distribution function $F_X(x) = \text{Prob}(X \leq x)$ with $x \in \mathbb{R}_+$ (in this paper we limit us to non-negative random variables).

• If one can write $F_X = \int_0^x f_X(y)dy$, one calls $f_X$ the density function of $X$.

• If the context is clear we will use $F(x)$ and $f(x)$ instead of $F_X(x)$ and $f_X(x)$.

• We note $F(x) = 1 - F_X(x)$.

For a random variable $X$ we define $0 \leq R$, $\forall R \in \mathbb{R}$. Thus $(X \land R) = \min( X, R)$.

It is the retained risk after an excess of loss (or stop loss) protection with priority $R$.

Note that $F_{(X \land R)}(x) = \text{Prob}(X \land R \leq x) = \begin{cases} F_X(x) & \text{if } R \geq x \\ 1 & \text{if } R < x \end{cases}$ and $f_{(X \land R)}(x) = \begin{cases} f_X(x) & \text{if } R \geq x \\ F_X(R) & \text{if } R < x \end{cases}$

• For a random variable $X(k)$ that depends of an exposure value $k \in \mathbb{R}_+$ (for example the insured value), we define the random variable $X_L(k) = \begin{cases} X(k) & \text{if } k \leq L \\ \frac{L}{k} X(k) & \text{if } k > L \end{cases}$.

It is the retained risk after a surplus protection with a retention line equal to $L \in \mathbb{R}_+$.

Note that

$F_{X_L(k)}(x) = \text{Prob}(X_L(k) \leq x) = \begin{cases} F_{X(k)}(x) & \text{if } L \geq k \\ F_{X(k)}\left(\frac{k}{L}\right) & \text{if } L < k \end{cases}$ and $f_{X_L(k)}(x) = \begin{cases} f_{X(k)}(x) & \text{if } L \geq k \\ \frac{k}{L} f_{X(k)}\left(\frac{k}{L}\right) & \text{if } L < k \end{cases}$

• Suppose a non-negative exposure value $K$ (for example sum insured) characterised by a density function $c(k)$ and a distribution function $C(k) = \int_{j=0}^{\infty} c(l)dl$.

Define $X$ as the mixture of $X(k)$ over $K$, the marginal distribution function of $X$ is equal to

$F_X(x) = \int_{k=0}^{\infty} c(k) \cdot F_{X(k)}(x)dk$ with density function $f_X(x) = \int_{k=0}^{\infty} c(k) \cdot f_{X(k)}(x)dk$. Analogous is

$F_{X_L}(x) = \int_{k=0}^{\infty} c(k) \cdot F_{X_L(k)}(x)dk = \int_{k=0}^{L} c(k) \cdot F_{X(k)}(x)dk + \int_{k=L}^{\infty} c(k) \cdot F_{X(k)}\left(\frac{k}{L}\right)dk$ and

$f_{X_L}(x) = \int_{k=0}^{\infty} c(k) \cdot f_{X_L(k)}(x)dk = \int_{k=0}^{L} c(k) \cdot f_{X(k)}(x)dk + \int_{k=L}^{\infty} c(k) \cdot \frac{k}{L} f_{X(k)}\left(\frac{k}{L}\right)dk$. 

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For each \( k \) we define \( Y(k) = \frac{X(k)}{k} \) and \( Y \) as the mixture of \( Y(k) \) over \( K \). Implying that
\[
F_Y(k)(y) = F_{X(k)}(k \cdot y) \quad \text{and} \quad f_Y(k)(y) = k \cdot f_{X(k)}(k \cdot y),
\]
which results in
\[
F_Y(y) = \int_{k=0}^{\infty} c(k) \cdot F_Y(k)(y) \, dk = \int_{k=0}^{\infty} c(k) \cdot F_{X(k)}(k \cdot y) \, dk \quad \text{and}
\]
\[
f_Y(y) = \int_{k=0}^{\infty} c(k) \cdot f_Y(k)(y) \, dk = \int_{k=0}^{\infty} c(k) \cdot f_{X(k)}(k \cdot y) \, dk.
\]

- If we combine an "excess of loss reinsurance with priority \( R \)" after a "surplus protection with line \( L \)" one becomes the retained risk \( (X_L \wedge R) = \min(R, X_L) \).

The corresponding distribution and density function have following expressions:

\[
F_{(X_L \wedge R)}(x) = \begin{cases} 
F_{X_L}(x) & \text{if } x \leq R \\
1 & \text{if } x > R
\end{cases}
\]

\[
f_{(X_L \wedge R)}(x) = \begin{cases} 
f_{X_L}(x) & \text{if } x < R \\
F_{X_L}(R) & \text{if } x = R \\
0 & \text{if } x > R
\end{cases}
\]

- Note that if for all \( k \) \( X(k) = k \cdot Z \) \( \text{or} \) \( Y(k) = Z \) previous formula reduces to
\[
F_{X(k)}(x) = F_Z \left( \frac{x}{k} \right) \quad \text{and} \quad f_{X(k)}(x) = f_Z \left( \frac{x}{k} \right) \cdot \frac{1}{k}.
\]
Which implies for \( x < R \):

\[
F_{(X_L \wedge R)}(x) = \int_{k=0}^{L} c(k) \cdot F_Z \left( \frac{x}{k} \right) \, dk + \int_{k=L}^{\infty} c(k) \cdot F_Z \left( \frac{x}{k} \right) \, dk = \int_{k=0}^{L} c(k) \cdot F_Z \left( \frac{x}{k} \right) \, dk + \int_{k=L}^{\infty} c(k) \cdot \frac{k}{L} f_Z \left( \frac{x}{k} \right) \, dk
\]

\[
= \int_{k=0}^{L} c(k) \cdot F_Z \left( \frac{x}{k} \right) \, dk + \frac{L}{k} f_Z \left( \frac{x}{L} \right) \cdot C(L)
\]

and

\[
f_{(X_L \wedge R)}(x) = \int_{k=0}^{L} c(k) \cdot f_Z \left( \frac{x}{k} \right) \, dk + \int_{k=L}^{\infty} c(k) \cdot f_Z \left( \frac{x}{k} \right) \cdot \frac{k}{L} \, dk = \int_{k=0}^{L} c(k) \cdot f_Z \left( \frac{x}{k} \right) \, dk + \frac{1}{L} \int_{k=L}^{\infty} c(k) \cdot f_Z \left( \frac{x}{L} \right) \, dk
\]

\[
= \int_{k=0}^{L} c(k) \cdot f_Z \left( \frac{x}{k} \right) \, dk + \frac{C(L)}{L} \cdot f_Z \left( \frac{x}{L} \right)
\]

Note that these formulas not explicitly depend on \( R \), which implies that they also correspond for a "surplus reinsurance without excess of loss protection" (the case \( R = \infty \)).

For \( x \geq R \) are the formulas straightforward and are left to the reader.
• A compound process is a random sum $S = \sum_{i=0}^{N} X_i$ with for all $i$ from 1 to $\infty$, $X_i \sim \text{iid}$, $X_0 = 0$ and $X$ independent on the non-negative random number $N$. We will call $S$ the aggregate loss and $X_i$ the (individual) loss.

For the first 2 moments of a compound process one has the following well-known formulas:

$$E(S) = E(N) \cdot E(X) \quad \text{and}$$

$$Var(S) = E(N) \cdot Var(X) + Var(N) \cdot E^2(X) = E(N) \cdot E(X^2) + \left[Var(N) - E(N)\right] \cdot E^2(X)$$

$$= E(N) \cdot \left[ E(X^2) + A \cdot E^2(X) \right] \geq 0 \quad \text{with} \quad A = \frac{Var(N) - E(N)}{E(N)} \in (-1, \infty)$$

(Eq 25)

$$Var(S) \geq E(N) \cdot Var(X) \quad \text{and} \quad Var(S) \geq Var(N) \cdot E^2(X)$$

We will use in this paper the abbreviation $Vrn(X) = E(X^2) + A \cdot E^2(X)$, which can be understood as a standard deviation of the portfolio normalised by the expected number of claims. Note that $Vrn(X) \geq Var(X)$.

### 7.2 Some basic derivations

For the application of the mean variance principle one needs to calculate the (partial) derivatives of the variance and expected value of retained risk with respect to priority (in case of excess of loss and stop loss reinsurance), retained line (in case of surplus protection) and retained share (in case of quota share).

To avoid a multiple derivation and to obtain the advantage of a general overview of these results we bring them together in one subsection.

We suppose that all the random variables will have very nice mathematical properties such that integrals can be changed of order, derivatives will exist and so on. We don’t want to minimise the importance of all these necessary conditions (and the research behind it), but like firstly to concentrate on the results one can become for the optimisation problem.

#### 7.2.1 Surplus & excess of loss

##### 7.2.1.1 General case

For an excess of loss protection after a surplus reinsurance we have that:

$$E(X_L \wedge R) = \int_{x=0}^{R} x \cdot f_{X_L}(x) dx + R \int_{x=R}^{\infty} f_{X_L}(x) dx$$

$$= \int_{x=0}^{R} x \cdot \left[ \int_{k=0}^{L} c(k) \cdot f_{X(k)}(x) dk + \int_{k=L}^{\infty} c(k) \cdot \frac{k}{L} \cdot f_{X(k)}(\frac{k}{L}x) dk \right] dx +$$

$$R \int_{x=R}^{\infty} \left[ \int_{k=0}^{L} c(k) \cdot f_{X(k)}(x) dk + \int_{k=L}^{\infty} c(k) \cdot \frac{k}{L} \cdot f_{X(k)}(\frac{k}{L}x) dk \right] dx$$

Changing the integrals gives:
\[ E(X_L \land R) = \sum_{k=0}^{k} c(k) \left[ \int_{x=0}^{x=R} f_{X(k)}(x)dx + R \int_{x=R} f_{X(k)}(x)dx \right] + \sum_{k=0}^{k} c(k) \left[ \int_{x=0}^{x} x \cdot \frac{k}{L} \cdot \frac{f_{X(k)}}{L}(x)dx + R \int_{x=0}^{x} \frac{k}{L} \cdot \left(\frac{k}{L}\right) dx \right] \]

With \( y = \frac{k \cdot x}{L} \) and \( r = \frac{R}{L} \), we conclude:

\[ E(X_L \land R) = \sum_{k=0}^{k} c(k) E\left[X(k) \land r \frac{L}{k}\right] + \sum_{k=0}^{k} c(k) \frac{L}{k} \left[ \int_{y=0}^{y=R} f_{X(k)}(y)dy + R \frac{k}{L} \int_{y=0}^{y=R} f_{X(k)}(y)dy \right] \]

\[ = \sum_{k=0}^{k} c(k) \cdot k \cdot E\left[Y(k) \land r \frac{L}{k}\right] + \sum_{k=0}^{k} c(k) \cdot E\left[Y(k) \land r \frac{L}{k}\right] \]

\[ \leq \sum_{k=0}^{k} c(k) \cdot k \cdot E\left[Y(k) \land r \frac{L}{k}\right] + \sum_{k=0}^{k} c(k) \cdot E\left[Y(k) \land r \frac{L}{k}\right] \]

\[ = L \cdot \sum_{k=0}^{L} c(k) \cdot E\left[Y(k) \land r \right] = L \cdot E\left[Y \land r \right] \]  \hspace{1cm} (Eq 26)

(*) follows from: for all \( k \leq L \) holds \( E\left[Y(k) \land r \frac{L}{k}\right] \leq \frac{L}{k} \cdot E\left[Y(k) \land r \right] \) because both functions have the same value for \( r = 0 \) but the derivative of the left-hand side function is never greater than the derivative of the right-hand side function \( (T_{y(k)}(r \frac{L}{k}) \leq \frac{L}{k} \cdot T_{y(k)}(r) \) because \( r \frac{L}{k} \geq r \).

Completely analogous one becomes that:

\[ E\left[(X_L \land R)^2\right] = \sum_{k=0}^{k} c(k) k^2 \left[ E\left[(Y(k) \land r \frac{L}{k})^2\right] + L^2 \sum_{k=0}^{k} c(k) E\left[(Y(k) \land r)^2\right] \right] \]  \hspace{1cm} (Eq 27)

For the derivatives of these formulas we need a less common known calculus result:

\[ \frac{\partial}{\partial L} \left[ \int_{k=0}^{L} g(k, h(L))dk \right] = g(L, h(L)) + \int_{k=0}^{L} g'(k, h(L)) \cdot h'(L)dk \]  \hspace{1cm} with \( g'(k, L) = \frac{\partial}{\partial L} g(k, L) \) and \( h'(L) = \frac{\partial}{\partial L} h(L) \)
\[
\frac{\partial E(X_L \land R)}{\partial L} = \frac{\partial}{\partial L} \left[ \int_{k=0}^{L} c(k) \cdot E[X(k) \land R] \, dk \right] + \frac{\partial}{\partial L} \left[ \sum_{k=L}^{\infty} c(k) \cdot E\left[ X(k) \land \frac{k R}{L} \right] \right]
\]

\[
= c(L) \cdot E[X(L) \land R] + \int_{k=L}^{\infty} c(k) \cdot E\left[ X(k) \land \frac{k R}{L} \right] \, dk - L \sum_{k=L}^{\infty} \frac{E\left[ X(L) \land \frac{R}{L} \right]}{L} c(L) \cdot \frac{R}{L} \, dk
\]

\[
L \int_{k=L}^{\infty} c(k) \cdot \frac{E\left[ X(k) \land \frac{k R}{L} \right]}{k} \, dk
\]

\[
= \int_{k=L}^{\infty} c(k) \cdot E\left[ X(k) \land \frac{k R}{L} \right] \, dk - \int_{k=L}^{\infty} c(k) \cdot F_{X(k)}\left( k \frac{R}{L} \right) \cdot \frac{R}{L} \, dk
\]

Conclusion

\[
\frac{\partial}{\partial L} E(X_L \land R) = \int_{k=L}^{\infty} c(k) \cdot E\left[ X(k) \land \frac{k R}{L} \right] \, dk - \int_{k=L}^{\infty} c(k) \cdot F_{X(k)}\left( k \frac{R}{L} \right) \, dk \geq 0 \text{ (with } r = \frac{R}{L})
\]

(Eq 28)

Completely analogous one becomes that:

\[
\frac{\partial}{\partial L} E[(X_L \land R)^2] = 2L \int_{k=L}^{\infty} c(k) \cdot \frac{E\left[ (X(k) \land \frac{k R}{L})^2 \right]}{k^2} \, dk - 2 \frac{R^2}{L^2} \int_{k=L}^{\infty} c(k) \cdot F_{X(k)}\left( k \frac{R}{L} \right) \, dk
\]

\[
= 2L \int_{k=L}^{\infty} c(k) \cdot \frac{E\left[ (X(k) \land \frac{k R}{L})^2 \right]}{k^2} \, dk - 2 \frac{R^2}{L^2} \int_{k=L}^{\infty} c(k) \cdot F_{X(k)}\left( k \frac{R}{L} \right) \, dk
\]

\[
= 2L \int_{k=L}^{\infty} c(k) \cdot \left[ \int_{y=0}^{\infty} f_{Y(k)}(y) \, dy \right] \, dk \geq 0 \text{ (with } r = \frac{R}{L}), \text{ or }
\]

\[
= 2L \left[ \int_{k=L}^{\infty} c(k) \cdot E\left[ (Y(k) \land r)^2 \right] \, dk - \int_{k=L}^{\infty} c(k) \cdot F_{Y(k)}(r) \cdot c(k) \, dk \right] \geq 0
\]

(Eq 29)
\[ \frac{\partial \text{Var} \left( \sum_{i=1}^{N} (X_L \land R)_{i} \right)}{\partial L} = E(N) \cdot \left[ \frac{\partial E \left( (X_L \land R)^2 \right)}{\partial L} + A \cdot \frac{\partial E^2 (X_L \land R)}{\partial L} \right] \]

\[ = E(N) \cdot \left[ \frac{\partial E \left( (X_L \land R)^2 \right)}{\partial L} + 2A \cdot E(X_L \land R) \cdot \frac{\partial E (X_L \land R)}{\partial L} \right] \]

\[ = 2E(N) \left[ L \cdot \int_{k=L}^{\infty} E \left( Y(k) \land r \right)^2 \cdot c(k)dk + A \cdot E(X_L \land R) \cdot \int_{k=L}^{\infty} E \left( Y(k) \land r \right) \cdot c(k)dk \right] - \]

\[ 2E(N) \cdot \left[ r \cdot L + A \cdot E(X_L \land R) \right] \cdot r \int_{k=L}^{\infty} \overline{F}_{y(k)} (r) \cdot c(k)dk \geq 0 \text{ if } A \geq 0 \quad (\text{Eq 30}) \]

For \( A \in [-1,0) \) is this derivative positive if (we denote \(|A|\) for the absolute value of \(A\))

\[ \int_{k=L}^{\infty} c(k) \left[ \int_{k=0}^{r} y^2 \cdot f_{y(k)} (y)dy \right] dk \geq |A| \cdot \frac{E(X_L \land R)}{L} \cdot \int_{k=L}^{\infty} c(k) \left[ \int_{k=0}^{r} y \cdot f_{y(k)} (y)dy \right] dk \quad (\text{Eq 31}) \]

From the equations (Eq 47), (Eq 48) and (Eq 51) we also known that:

\[ \frac{\partial}{\partial r} E(X_L \land R) = \overline{F}_{x_k} (R) \geq 0, \quad \frac{\partial}{\partial r} E \left[ (X_L \land R)^2 \right] = 2R \cdot \overline{F}_{x_k} (R) \geq 0 \]

\[ \frac{\partial \text{Var} \left( \sum_{i=1}^{N} (X_L \land R)_{i} \right)}{\partial R} = 2E(N) \cdot \overline{F}_{x_k} (R) \cdot \left[ R + A \cdot E(X_L \land R) \right] \geq 0 \quad (\text{Eq 32}) \]

7.2.1.2 Special case: \( X(k) = k \cdot Z \)

Note that for all \( k \quad X(k) = k \cdot Z \) or \( Y(k) = Z \) previous formulas reduce to \((r = \frac{R}{L})\)

\[ E(X_L \land R) = \int_{k=0}^{L} k \cdot c(k) \cdot E[Z \land R/k]dk + L \cdot E[Z \land r] \cdot \overline{C}(L) \]

\[ \leq \left( \ast \right) \int_{k=0}^{L} k \cdot c(k) \cdot \frac{L}{k} E[Z \land r]dk + L \cdot E[Z \land r] \cdot \overline{C}(L) = L \cdot E[Z \land r] \quad (\text{Eq 33}) \]

For \((\ast)\) see (Eq 26)

\[ E \left( (X_L \land R)^2 \right) = \int_{k=0}^{L} c(k) \cdot k^2 \cdot E \left[ (Z \land R/k)^2 \right]dk + L^2 \cdot E \left[ (Z \land r)^2 \right] \cdot \overline{C}(L) \quad (\text{Eq 34}) \]

\[ \frac{\partial}{\partial L} E(X_L \land R) = \left[ E[Z \land r] - r \cdot \overline{F}_z (r) \right] \cdot \overline{C}(L) = \overline{C}(L) \int_{z=0}^{r} z \cdot f_z (z)dz \quad (\text{Eq 35}) \]

\[ \frac{\partial}{\partial L} E \left( (X_L \land R)^2 \right) = 2L \cdot \left[ E \left( (Z \land r)^2 \right) - r^2 \cdot \overline{F}_z (r) \right] \cdot \overline{C}(L) = 2L \cdot \overline{C}(L) \cdot \int_{z=0}^{r} z^2 \cdot f_z (z)dz \quad (\text{Eq 36}) \]
\[ \frac{\partial}{\partial R} E(X_L \land R) = \overline{F}_{X_L}(R) = \int_{k=0}^{L} c(k) \cdot F_Z\left(\frac{R}{k}\right)dk + \frac{R}{L} \cdot \overline{C}(L) \] and \[ \frac{\partial}{\partial R} E\left[(X_L \land R)^2\right] = 2R \cdot \overline{F}_{X_L}(R) \]

\[ \frac{\partial \text{Var}\left(\sum_{i=1}^{N}(X_L \land R)_i\right)}{\partial L} = 2E(N) \cdot \overline{C}(L) \cdot \left[ L \cdot \int_{z=0}^{r} z^2 \cdot f_Z(z)dz + A \cdot E(X_L \land R) \cdot \int_{z=0}^{r} z \cdot f_Z(z)dz \right] \]

\[ \frac{\partial \text{Var}\left(\sum_{i=1}^{N}(X_L \land R)_i\right)}{\partial R} = 2E(N) \cdot \left[ \int_{k=0}^{r} c(k) \cdot F_Z\left(\frac{R}{k}\right)dk + F_Z(r) \cdot \overline{C}(L) \right] \cdot \left[R + A \cdot E(X_L \land R)\right] \]

7.2.1.3 Optimal solution

As mentioned before (see paragraph 3) the Lagrange equations become \( \frac{\partial}{\partial W} \Psi(L,R) = 0 \) with \( W=L \) or \( R \), and results in:

\[ \frac{\partial}{\partial R} \Psi(L,R) = 2E(N) \cdot \left[R + A \cdot E(X_L \land R)\right] \overline{F}_{X_L}(R) + \frac{1}{\mu} \cdot E(N) \cdot \left\{-\overline{F}_{X_L}(R) + (1+\lambda_x) \cdot \overline{F}_{X_L}(R)\right\} = 0 \Rightarrow \]

\[ R + A \cdot E(X_L \land R) = -\frac{1}{2\mu} \lambda_x \]  \hspace{1cm} (Eq 37)

\[ \frac{\partial}{\partial L} \Psi(L,R) = E(N) \cdot \left[ \frac{\partial E\left[(X_L \land R)^2\right]}{\partial L} + 2A \cdot E(X_L \land R) \frac{\partial E(X_L \land R)}{\partial L} \right] + \]

\[ \frac{1}{\mu} \cdot E(N) \cdot \left\{- \frac{\partial E(X_L \land R)}{\partial L} + (1+\lambda_x) \frac{\partial E(X_L \land R)}{\partial L} - (1+\lambda_x) \cdot \frac{\partial E(X_L \land R)}{\partial L} \right\} = 0 \Rightarrow \]

\[ \frac{\partial E\left[(X_L \land R)^2\right]}{\partial L} + 2A \cdot E(X_L \land R) \frac{\partial E(X_L \land R)}{\partial L} = -\frac{1}{\mu} \left\{ \lambda_x \frac{\partial E(X_L \land R)}{\partial L} - (\lambda_x - \lambda_y) \frac{\partial E(X_L \land R)}{\partial L} \right\} \Rightarrow \]

Denoting \( Q = \frac{\lambda_x - \lambda_y}{\lambda_x} \in (0,1) \)

\[ L \int_{k=L}^{\infty} \frac{c(k)}{k^2} \left[ \int_{y=0}^{r} x^2 \cdot f_{X(k)}(x)dx \right]dk + A \cdot E(X_L \land R) \int_{k=L}^{\infty} \frac{c(k)}{k} \left[ \int_{y=0}^{r} x \cdot f_{X(k)}(x)dx \right]dk = - \]

\[ \int_{k=L}^{\infty} \frac{c(k)}{k} \left[ \int_{y=0}^{r} x \cdot f_{X(k)}(x)dx \right]dk - Q \int_{k=L}^{\infty} \frac{E[X(k)]c(k)}{k}dk \]  \hspace{1cm} (Eq 38)

It becomes also clear that the random variable \( X(k) \) divided by its exposure value \( k \) plays an important role.

We define \( Y(k) = \frac{X(k)}{k} \) for each \( k>0 \) and \( Y_L = \frac{X_L}{L} \) for each \( L>0 \).
Using the notations $CB, CB1, CB2, CC, CC1$ and $CD$ defined as follows

\[
CB = \int_{k=L}^{\infty} \frac{c(k)}{k^2} \left[ \int_{x=0}^{k} x^2 \cdot f_{X(k)}(x) \, dx \right] \, dk = \int_{k=L}^{\infty} c(k) \left[ \int_{y=0}^{k} y^2 \cdot f_{Y(k)}(y) \, dy \right] \, dk
\]

\[
= \int_{k=L}^{\infty} c(k)E[(Y(k) \land r)^2] \, dk - r^2 \int_{k=L}^{\infty} F_{Y(k)}(r) \cdot c(k) \, dk \quad = CB1 + CB2
\]

\[
CC = \int_{k=L}^{\infty} \frac{c(k)}{k} \left[ \int_{x=0}^{k} x \cdot f_{X(k)}(x) \, dx \right] \, dk = \int_{k=L}^{\infty} c(k)E[Y(k) \land r] \, dk - r \int_{k=L}^{\infty} F_{Y(k)}(r) \cdot c(k) \, dk \quad = CC1 + \frac{CB2}{r}
\]

\[
CD = \int_{k=L}^{\infty} \frac{E[X(k)]}{k} c(k) \, dk = \int_{k=L}^{\infty} E[Y(k)]c(k) \, dk
\]

We can rewrite (Eq 37) like $L \cdot CB + A \cdot E(X_L \land R) \cdot CC = -\frac{1}{2\mu} \lambda \{CC - Q \cdot CD\}$ and combining with (Eq 38) gives:

\[
L \cdot CB + A \cdot E(X_L \land R) \cdot CC = \left[ R + A \cdot E(X_L \land R) \left[ CC - Q \cdot CD \right] \right] \quad \Rightarrow
\]

\[
L \cdot CB1 + A \cdot E(X_L \land R) \cdot CC1 + L \cdot CB2 + A \cdot E(X_L \land R) \quad = \frac{CB2}{r}
\]

\[
\left[ R + A \cdot E(X_L \land R) \right] \cdot \left[ CC1 - Q \cdot CD \right] + \left[ R + A \cdot E(X_L \land R) \right] \frac{CB2}{r}
\]

This reduces nicely to $L \cdot CB1 + A \cdot E(X_L \land R) \cdot CC1$ or equivalently

\[
r + A \cdot E(Y_L \land r) = \int_{k=L}^{\infty} c(k)E[(Y(k) \land r)^2] \, dk + A \cdot E(Y_L \land r) \cdot \int_{k=L}^{\infty} c(k)E[Y(k) \land r] \, dk
\]

\[
+ \int_{k=L}^{\infty} c(k)E[Y(k) \land r] \, dk - Q \int_{k=L}^{\infty} E[Y(k)]c(k) \, dk
\]

\[
= \int_{k=L}^{\infty} \left\{ E[(Y(k) \land r)^2] + A \cdot E(Y_L \land r) \cdot E[Y(k) \land r] \right\} c(k) \, dk
\]

\[
\int_{k=L}^{\infty} \left\{ E[Y(k) \land r] - Q \cdot E[Y(k)] \right\} c(k) \, dk
\]

\[
(Eq 39)
\]

7.2.2 Surplus & individual loss

Based on (Eq 26) and (Eq 27) with $R = \infty$ we become that the first 2 moments for the individual loss after a surplus protection:

\[
E(X_L) = \int_{k=0}^{L} k \cdot E[Y(k)] \cdot c(k) \, dk + L \cdot E[Y(k)] c(k) \, dk \leq L \cdot E(Y)
\]

(Eq 40)

This implies for $L = \infty$ that $E(X) = \int_{k=0}^{\infty} E[X(k)] \cdot c(k) \, dk = \int_{k=0}^{\infty} k \cdot E[Y(k)] \cdot c(k) \, dk$.
\[ E(X^2_k) = \int_{k=0}^{L} k^2 \cdot E[Y^2(k)] \cdot c(k) \, dk + L^2 \int_{k=L}^{\infty} E[Y^2(k)] \cdot c(k) \, dk \leq L^2 \cdot E(Y^2) \]  
(Eq 41)

This implies for \( L = \infty \) that \( E(X^2) = \int_{k=0}^{\infty} E[X^2(k)] \cdot c(k) \, dk = \int_{k=0}^{\infty} k^2 \cdot E[Y(k)] \cdot c(k) \, dk \).

Their derivatives follow from (Eq 28), (Eq 29), (Eq 30):

\[ \frac{\partial E(X_L)}{\partial L} = \int_{k=L}^{\infty} E[Y(k)] \cdot c(k) \, dk \geq 0 \quad \text{and} \quad \frac{\partial E(X^2_L)}{\partial L} = 2L \int_{k=L}^{\infty} E[Y^2(k)] \cdot c(k) \, dk \geq 0 \]  
(Eq 42)

\[ \frac{\partial \text{Var} \left( \sum_{i=1}^{N} (X_{L_i}) \right)}{\partial L} = 2E(N) \cdot \left[ \int_{k=L}^{\infty} \left\{ L \cdot E[Y^2(k)] + A \cdot E(X_L) \cdot E[Y(k)] \right\} c(k) \, dk \right] \geq 0 \quad \text{if} \quad A \geq 0 \]

The general condition to have a positive derivative for \( A \in [-1, 0) \) is given by

\[ \int_{k=L}^{\infty} E[Y^2(k)] \cdot c(k) \, dk \geq \frac{|A|}{L} \quad \text{(with} \quad |A| \quad \text{denoting the absolute value of} \quad A) \]  
(Eq 43)

7.2.2.1 **Special case:** \( X(k) = k \cdot Z \)

Note that for all \( k \) \( X(k) = k \cdot Z \) or \( Y(k) = Z \) previous formulas reduce to

\[ E(X_L) = E(Z) \cdot E(K \land L) \quad \text{and} \quad E(X^2_L) = E(Z^2) \cdot E[(K \land L)^2] \]  
(Eq 44)

\[ \text{Var}(X_L) = E(Z^2) \cdot E[(K \land L)^2] - E^2(Z) \cdot E^2(K \land L) \]  
(Eq 45)

\[ \frac{\partial E(X_L)}{\partial L} = E(Z) \cdot \overline{C}(L) \geq 0 \quad \text{and} \quad \frac{\partial E(X^2_L)}{\partial L} = 2L \cdot E(Z^2) \cdot \overline{C}(L) \geq 0 \]  
(Eq 46)

\[ \frac{\partial \text{Var} \left( \sum_{i=1}^{N} (X_{L_i}) \right)}{\partial L} = 2 \cdot \overline{C}(L) \cdot E(N) \cdot \left[ L \cdot E(Z^2) + A \cdot E^2(Z) \cdot E(K \land L) \right] \quad \text{if} \quad A \in [-1, \infty) \]

\[ \geq 2 \cdot \overline{C}(L) \cdot E(N) \cdot \left[ L \cdot E(Z^2) - E^2(Z) \cdot E(K \land L) \right] \]

\[ \geq 2 \cdot \overline{C}(L) \cdot E(N) \cdot \left[ L \cdot E(Z^2) - E^2(Z) \cdot L \right] = 2 \cdot \overline{C}(L) \cdot E(N) \cdot L \cdot \text{Var}(Z) \geq 0 \]

7.2.3 **Stop loss and excess of loss**

For an excess of loss or stop loss we have that:

\[ E(X \land R) = \int_{0}^{R} x \cdot dF(x) + R \int_{R}^{\infty} dF(x) = \int_{0}^{R} x \cdot dF(x) + R \cdot \overline{F}(R) \leq R \quad \text{and} \quad E(X \land R) \geq R \cdot \overline{F}(R) \]
\[
\frac{\partial E(X \land R)}{\partial R} = R \cdot f(R) - R \cdot f(R) + \overline{F}(R) = \overline{F}(R) \geq 0 \quad \text{(Eq 47)}
\]

\[
E(X \land R)^2 = \int_0^R x^2 dF(x) + R^2 \int_0^\infty dF(x) = \int_0^R x^2 dF(x) + R^2 \cdot \overline{F}(R) \leq R^2
\]

\[
\frac{\partial E(X \land R)^2}{\partial R} = R^2 \cdot f(R) - R^2 \cdot f(R) + 2R \cdot \overline{F}(R) = 2R \cdot \overline{F}(R) \geq 2R \geq 0 \quad \text{(Eq 48)}
\]

\[
\frac{\partial \text{Var}(X \land R)}{\partial R} = \frac{\partial E(X \land R)^2}{\partial R} - \frac{\partial E^2(X \land R)}{\partial R} = 2\left[R - E(X \land R)\right] \cdot \overline{F}(R) \geq 0 \quad \text{(Eq 49)}
\]

\[
\text{Var}(X \land R) = E\left[(X \land R)^2\right] - E^2(X \land R) \leq \left[R - E(X \land R)\right] \cdot \overline{F}(R), \quad E(X \land R) \leq \left[R - R \cdot \overline{F}(R)\right] \cdot \overline{F}(X \land R) = R \cdot F(R) \cdot \overline{F}(X \land R) \leq R^2 \cdot F(R) \leq R^2
\]

\[
\frac{\partial \text{Var}\left(\sum_{i=1}^{N}(X \land R)_i\right)}{\partial R} = E(N) \cdot \left[\frac{\partial E\left[(X \land R)^2\right]}{\partial R} + A \cdot \frac{\partial E^2(X \land R)}{\partial R}\right] \quad \text{(Eq 51)}
\]

which implies that the variance of a compound excess of loss contract is (strictly) increasing because \(A \geq -1\).

If the distribution \(X_L\), a mixture is of \(X_j(k)\), with \(X_L \in \{0, \infty\}\), over the exposure distribution \(K\), we refer to the corresponding equations in section 7.2.1.

### 8 References

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