Debt stabilizing fiscal rules

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Abstract

This paper investigates properties of a state-contingent debt targeting rule which links stabilizing budgetary adjustments around a target level of long-run debt to the state of the economy. Considering a range of fiscal instruments, the paper shows that the size of steady-state debt is a key determinant of whether such a rule can be implemented under all available instruments. The key theoretical result of the paper states that the number of instruments which can be used to implement such a rule declines in the level of steady-state debt. This finding is of particular relevance in the context of a monetary union with decentralized fiscal policies. Depending on the level of long-run debt, there might be a conflict between a common fiscal framework and the unrestricted choice of fiscal policy instruments at the national level.

Keywords: Fiscal regimes, Overlapping generations

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1 Introduction

Unstable government debt dynamics can typically be corrected by appropriate budgetary adjustments. Moreover, to achieve the needed corrections a government can typically adjust a broad range of fiscal instruments, like government spending, public transfers, or various taxes. Given this multiplicity of fiscal instruments, this paper starts out from the idea that there are two different ways to conceptualize state-contingent fiscal rules which stabilize government debt dynamics around a certain long-run level of debt. First, for a particular instrument one can think of rules which establish a link between stabilizing variations of the instrument and the state of the economy. For a particular specification of such a rule it is then possible to derive the implied sequence of budgetary adjustments. Second, one can think of broader rules which establish a link between budgetary adjustments and the state of the economy. This reasoning leads to the question under which instruments a particular specification of such a rule can be implemented. Whether it is preferable to condition the path of budgetary adjustments on the state-contingent path of a particular instrument (or a combination of various instruments) or, alternatively, to condition the choice of the instrument on a particular state-contingent path of budgetary adjustments depends on the context at hand. This paper argues that the second line of reasoning is particularly relevant for the design of fiscal rules in a monetary union with decentralized fiscal policies. Specifically, it can be used to see that, depending on the target level of government debt, conflicts may arise between a common fiscal framework which tracks deficit developments and the unrestricted choice of fiscal policy instruments at the national level.

To see intuitively, why a debt targeting rule, when expressed in terms of budgetary adjustments, may not always be implementable under all instruments consider, for simple illustration, a government which can generate surpluses by reducing government expenditures or alternatively by raising wage income taxes. In a life-cycle framework one can well imagine that the second measure will decrease private sector savings, while the first measure may leave savings constant. Accordingly, for any given fiscal consolidation requirement, the crowding out of private sector investments is likely to be different under the two instruments, implying that the state of the economy is likely to evolve differently under the instruments. As a result, it is a priori not clear whether for a given specification of such a debt targeting rule its implementability under both instruments will be ensured.

To further explore this issue, this paper considers a small and fully tractable model which distinguishes between three distinct fiscal instruments in the government’s flow budget constraint, namely government consumption, transfer payments and a wage income tax. The analysis is based on an overlapping generations economy with government debt dynamics. To operationalize the notion of unstable government debt dynamics, the paper identifies steady states which are characterized by
non-negative levels of government debt and which are locally unstable under the specific assumption of a permanently balanced primary budget. However, the economy can be stabilized at the corresponding steady state levels if one allows for appropriate budgetary adjustments. To this end, we consider a simple debt targeting rule which links the adjustments of the primary balance to the two state variables of the model, physical capital and real government bonds, with appropriate feedback coefficients. We derive first for all three instruments the range of feedback coefficients which stabilize government debt at the target value. Then, to establish whether the debt targeting rule can be implemented under all instruments, we check whether the ranges of stabilizing feedback coefficients established for the three instruments do overlap. The paper shows that the answer to this question depends on the level of long-run debt around which the economy is stabilized. Specifically, the key theoretical result of the paper states that the number of instruments which can be used to implement such a rule declines in the level of steady-state debt.

Essentially, this result reflects that our model economy consists of two parts: the budget constraint of the government and a block which summarizes all the remaining private sector activities in the economy. By construction, the source of instability is confined to the first part, while the instruments which can be used to achieve the required fiscal surpluses affect the second part through different margins. The level of steady-state debt determines the relative importance of these margins within the set of intertemporal equilibrium conditions. Under a low level of steady-state debt, these margins carry low weights, making it possible that the debt targeting rule can be implemented with a common set of feedback coefficients under all three instruments. With high debt, however, these margins gain importance, and, as debt increases, the instrument-specific stabilization profiles become so diverse that the debt targeting rule can no longer be implemented under all three instruments.

In related literature, Schmitt-Grohé and Uribe (1997) and Guo and Harrison (2004) show that for a given fiscal rule (in their context: a balanced-budget rule) different fiscal instruments can lead to different stability properties of long-run equilibria. Specifically, depending on whether budget balance is achieved by distortionary income taxes or government spending adjustments, equilibria can be locally unique or indeterminate. Our paper shares with these two papers the descriptive nature of the fiscal rule.¹ Implementation issues of fiscal policy have also been addressed in a large number of papers which explicitly solve for optimal fiscal (and monetary) policies.² These studies are typically conducted in the context of model economies with Ramsey-
type infinitely lived agents. In such modelling environments, however, there is little conceptual agreement about the optimal target level of long-run debt itself. Reflecting this feature, recent studies by Kollmann (2004) and Schmitt-Grohé and Uribe (2004), for example, consider only a restricted set of optimal policies around a certain pre-specified level of debt. By contrast, the long-run target levels of government debt analyzed in our overlapping generations structure have a simple normative foundation because of dynamic efficiency considerations.

We think that one particularly interesting application of our main result is given by monetary unions in which member states remain responsible for their national and sub-national budgetary policies, subject to the provisions of a common fiscal framework that are needed to keep free-riding incentives at the national level in check. Evidently, the European Monetary Union is a good example for this, since the Treaty and the Stability and Growth Pact constitute a rule-based fiscal framework that sets certain limits to deficits and debt levels and strengthens multilateral budgetary surveillance. In general, in EMU the distribution of responsibilities between the union level and the member states is governed by the two fundamental principles of subsidiarity and proportionality. Applied to the field of fiscal policy these principles establish a clear distinction between the role of the union and that of member states. In principle, member states are responsible for their national budgetary policies, while the common fiscal framework of budgetary surveillance, which is implemented at the level of the union, monitors how broad fiscal indicators, like the deficits of member states, react to the state of the economy. Moreover, whenever corrective fiscal policy measures are needed, the framework respects, in principle, national preferences with respect to the implementation of such measures. Evidently, the broad modelling assumptions maintained in this paper cannot fully capture further institutional details which characterize this particular arrangement.

3In particular, in the framework of Aiyagari et al. (2002) the optimal long-run level of government debt is shown to be negative because of the non-distortionary nature of the interest income that a government receives in such a constellation. Benefits of positive government debt, like the loosening of private sector borrowing constraints because of an enhanced liquidity position, are discussed in Aiyagari and McGrattan (1998). Costs and benefits of long-run government debt levels are also discussed in Martin (2004).

4For a similar assumption in a model dealing with optimal fiscal rules in the context of a monetary union, see, for example, Lambertini (2004). Moreover, the studies by Schmitt-Grohé and Uribe as well as by Kollmann show that simple fiscal feedback rules may well have welfare properties similar to those which one obtains from the optimizing Ramsey-programs.


6These principles are laid down in Article 5 of the Treaty (and Article I-9 of the draft Constitution). According to the principle of subsidiarity, the Union shall act only if and insofar as the objectives of the intended action cannot be sufficiently achieved by the Member States. According to the principle of proportionality, any action by the Union shall not go beyond what is necessary to achieve the objectives of the Treaty.
Yet, irrespective of these details, the analysis of this paper clearly indicates that a sufficiently low level of average debt facilitates the smooth functioning of any carefully balanced arrangement of this type. By contrast, at high levels of average debt conflicts may arise between the provisions of a common fiscal framework and the unrestricted choice of fiscal policy instruments at the national level.

The paper is organized as follows. As a particularly tractable starting point, Section 2 presents a Diamond-type overlapping generations model with an exogenous labour supply, enriched with a government sector and public debt. The model allows for three fiscal instruments, namely government consumption, lump-sum taxes on young agents and lump-sum transfers to old agents. For further reference, Section 2 reproduces from the literature key features of two types of steady states which are locally unstable under a permanently balanced primary budget: i) underaccumulation steady states with zero government debt and ii) golden rule steady states with positive debt. Section 3 introduces the debt targeting rule and derives the main results of the paper. Specifically, it is shown that at steady states with zero debt implementability of the debt targeting rule is ensured under all three instruments. By contrast, at steady states with positive debt this is no longer true and we establish that the number of instruments which can be used to stabilize debt dynamics via the debt targeting rule declines in the level of steady-state debt. Section 4 establishes the robustness of the main results of Section 3 along two dimensions. First, we consider two alternative, broad state-contingent debt targeting rules which are no longer expressed in terms of adjustments of the primary balance. Specifically, it is shown that the results of Section 3 remain unaffected if the debt targeting rule imposes instead state-contingent restrictions on the path of the overall deficit or, alternatively, directly on the path of newly emitted debt. Second, we allow for an endogenous labour supply and distortionary wage income taxation and show that this modification does not change the basic results. Section 5 offers conclusions. Proofs and some technical issues are delegated to two Appendices at the end of the paper.

2 The model with exogenous labour supply

For simple tractability the first part of the paper is based on a version of a Diamond-type overlapping generations economy with exogenous labour supply and lump-sum taxes and transfers.

Problem of the representative agent

In period $t$, the economy is populated by a large number $N_t$ of young agents and $N_{t-1}$ of old agents. Each agent lives for two period and has a time-invariant, fixed labour supply $l = 1$ when being young and a zero labour supply when being old.
The population grows at the constant rate $n > 0$, i.e. $N_t = (1 + n) \cdot N_{t-1}$. Let preferences of the representative agent born in period $t$ be given by

$$U(c_t, d_{t+1}),$$

where $c_t$ and $d_{t+1}$ denote first-period and second-period consumption, respectively.

\textbf{(A 1)} The function $U(c, d)$ is twice continuously differentiable, strictly increasing, strictly quasi-concave and satisfies for all $\overline{c}, \overline{d} > 0$, $\lim_{c \to 0} U_c(c, \overline{d}) \to \infty$ and $\lim_{d \to 0} U_d(\overline{c}, d) \to \infty$.

In any period $t$, agents take the wage rate $(w_t)$ and the return factor $R_{t+1}$ on savings $(s_t)$ as given. There exists a tax-transfer-system such that young agents pay lump-sum taxes $\eta_t > 0$, while they receive lump-sum transfers $\theta_{t+1}$ when being old.\footnote{We do not make any explicit sign restrictions regarding the second-period lump-sum payment $\theta_{t+1}$. Strictly speaking, the term ‘tax-transfer’-system would refer only to a scenario with $\theta_{t+1} > 0$.} This leads to the pair of budget constraints

$$w_t - \eta_t = c_t + s_t$$
$$d_{t+1} = R_{t+1}s_t + \theta_{t+1},$$

which can be used to rewrite the objective as:

$$U(w_t - \eta_t - s_t, R_{t+1}s_t + \theta_{t+1}).$$

The optimal choice of savings is characterized by the first-order condition

$$U_1 = R_{t+1}U_2.$$

To characterize the savings decision of agents, we refer to the well-investigated Diamond model without second-period transfers (i.e. $\theta_{t+1} = 0$) and assume $w - \eta > 0$. Then, according to (A 1), there exists the savings function

$$s(w - \eta, R) = \arg \max U(w - \eta - s, Rs), \quad (1)$$

with $s(w - \eta, R) : R_{++} \times R_{++} \to R_{++}$ being continuously differentiable. In order to extend (1) to a situation with $\theta_{t+1} \neq 0$ it is assumed that the life-cycle income is positive, i.e. $w - \eta + \frac{\theta}{R} > 0$. Then, savings will be given by

$$s_t = s(w_t - \eta_t + \frac{\theta_{t+1}}{R_{t+1}}, R_{t+1}) - \frac{\theta_{t+1}}{R_{t+1}}.$$
and \( s_t \) satisfies \(-\frac{\theta_{t+1}}{n_{t+1}} < s_t < w_t - \eta_t \) to ensure non-negative consumption in both periods.\(^8\) Finally, to impose further structure on the function \( s(w, R) \), we make the customary assumption:

(A 2) Consider again \( U(c, d) \) and assume that consumption goods are normal and gross substitutes. Then, \( 0 < s_w < 1 \) and \( s_R \geq 0 \).

**Production**

It is assumed that there exists a larger number of competitive firms with access to a standard neoclassical technology \( F(K_t, L_t) \), where \( K \) and \( L \) denote the aggregate levels of physical capital and labour, respectively.

(A 3) The function \( F(K, L) : R_{++} \times R_{++} \to R_{++} \) is positive valued, twice continuously differentiable, homogenous of degree 1, increasing and satisfies \( F_{KK}(K, L) < 0 \).

Firms are price takers in input and output markets. In a competitive equilibrium, labour market clearing requires \( L_t = N_t \). Let \( k_t = K_t/N_t \) denote the capital stock per young agent, giving rise to the familiar pair of first-order conditions

\[
R_t = 1 - \delta + F_K(k_t, 1) = R(k_t) \tag{2}
\]

\[
w_t = F_L(k_t, 1) = w(k_t), \tag{3}
\]

with \( \delta \) denoting the depreciation rate on capital. According to (2) and (3), the equilibrium return rates of the two production factors depend only on the equilibrium capital intensity and change along the factor price frontier with \( R'(k_t) = F_{KK}(k_t, 1) < 0 \) and \( w'(k_t) = F_{LK}(k_t, 1) > 0 \).

**Government**

In the representative period \( t \), the government consumes an amount \( G_t \) of aggregate output which does not affect the utility of consumers.\(^9\) Let

\[
\Pi_t = N_t \eta_t - N_{t-1} \theta_t - G_t
\]

denote the primary surplus. In per capita form this reads as

\[
\pi_t = \eta_t - \frac{\theta_t}{1 + n} - g_t
\]

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\(^8\)For a more detailed discussion of the savings problem under second-period lump-sum payments, see de la Croix and Michel p. 130 f.

\(^9\)If publicly provided public goods enter the utility of the representative consumer in an additively separable manner, they do not affect the consumer’s saving decision. Our analysis would still hold under this assumption.
where $g_t$ and $\pi_t$ denote government consumption and the primary surplus per young individual, respectively.

It is assumed that agents perceive investments in physical capital and government bonds (in real terms) as perfect substitutes with identical return factor $R_t$. Then, the flow budget constraint of the government, expressed per young agent, reads as

$$(1 + n) b_{t+1} = R(k_t) b_t - \pi_t.$$  

**Intertemporal equilibrium conditions**

In sum, we obtain a version of the intertemporal equilibrium conditions of the Diamond-model, modified for the existence of a simple tax-transfer system and the possibility of a primary balance that does not have to be balanced in every period

$$(1 + n)(k_{t+1} + b_{t+1}) = s_t = s(w(k_t) - \eta_t + \frac{\theta_{t+1}}{R(k_{t+1})} R(k_{t+1})) - \frac{\theta_{t+1}}{R(k_{t+1})} \quad (4)$$

$$(1 + n)b_{t+1} = R(k_t) b_t - \pi_t \quad (5)$$

$$\pi_t = \eta_t - \frac{\theta_t}{1 + n} - g_t. \quad (6)$$

**Initial conditions**

In each period $t$, the state of the economy is summarized by the pair $(b_t, k_t)$, denoting the beginning-of-period per capita values of the capital stock and of real government bond holdings which are determined by past investment decisions taken in period $t-1$. Hence, when we subsequently classify the dynamic behaviour of the system (4)-(6) under various fiscal closures, it is natural to assume that dynamics are characterized by two initial conditions, $b_0$ and $k_0$.\(^{(10)}\)

**Dynamics under a permanently balanced primary budget**

The equations (4)-(6), without further restrictions, allow for a rich set of dynamic equilibria. In the remainder of this paper, however, we focus on the local stability behaviour of the system around steady states which have the particular feature of a balanced primary budget (i.e. $\pi = 0$). Moreover, to set the stage for a meaningful discussion of stabilizing off-steady-state adjustments in the primary balance in Section 3, we establish benchmark steady states of (4)-(6) which, assuming $b_0 \neq b$ and $k_0 \neq k$, are unstable if the primary budget is permanently balanced, i.e. if $\pi_t \equiv 0$ for all $t$. Specifically, consider a stationary tax-transfer system with $g \equiv \eta - \frac{\theta}{1 + n} > 0$.

\(^{(10)}\)Note, however, that there is also a branch of the literature which stresses the role of bubbles in closely related models and treats real government debt as a jumping variable; see Tirole (1985). More recently, the treatment of real government debt as a jumping variable plays also a key role in the logic of the fiscal theory of the price level, as summarized, for example, in Woodford (2001).
leading to the two-dimensional dynamic system in $k_t$ and $b_t$

\[
(1 + n)(k_{t+1} + b_{t+1}) = s_t = s(w(k_t) - \eta + \frac{\theta}{R(k_{t+1})}, R(k_{t+1})) - \frac{\theta}{R(k_{t+1})}
\]  
\[
(1 + n)b_{t+1} = R(k_t)b_t.
\]  

Using a first-order approximation, dynamics around steady states of (7) and (8) evolve according to

\[
A_1 \cdot dk_{t+1} + (1 + n) \cdot db_{t+1} = A_2 \cdot dk_t
\]

\[
(1 + n) \cdot db_{t+1} = R'(k)b \cdot dk_t + R(k) \cdot db_t,
\]  
with:

\[
A_1 = 1 + n - R'(k) \cdot [s_R + (1 - s_w) \frac{\theta}{R(k)^2}]
\]

\[
A_2 = s_w w'(k) > 0
\]

Existence and stability conditions of steady states of (7) and (8) have been widely discussed in the literature. In particular, under mild assumptions the system is associated with two distinct types of steady states which are unstable under a permanently balanced primary budget: i) steady states with zero debt and underaccumulation ($k > 0, b = 0, R(k) > 1 + n$) and ii) golden rule steady states with positive debt ($k > 0, b > 0, R(k) = 1 + n$). To this end, we make the assumption

(A 4) There exist steady states of (7) and (8) with $k > 0$ and $b \geq 0$, satisfying $A_2 < A_1$.

Remark: Assumption (A 4) is not very restrictive. For illustration, assume first $g = \eta = \theta = 0$. Then, if assumptions (A1)-(A 3) are satisfied and, for example, the aggregate production function is of Cobb-Douglas type, there exists a unique steady state with $k > 0$ and $b = 0$, satisfying $A_2 < A_1$.\textsuperscript{11} Moreover, if at this steady state $R(k) < 1 + n$, there exists a golden rule steady state with $k > 0, b > 0$ and $R(k) = 1 + n$, also satisfying $A_2 < A_1$. If $g = \eta > 0$ and $\theta = 0$, this reasoning can be extended as long as $g$ is smaller than some positive bound $\overline{g}$. Finally, by continuity, if $g \equiv \eta - \frac{\theta}{1 + n} > 0$ and $\theta \neq 0$, steady states continue to exist as long as $\theta$ is sufficiently small.\textsuperscript{12}

\textsuperscript{11}If the production function is of the more general CES-type this reasoning extends to the case of an elasticity of substitution larger than one. If the elasticity is less than one, there are zero or two steady states with $k > 0$ and $b = 0$. In the latter case, the high activity steady state satisfies $A_2 < A_1$.

\textsuperscript{12}For a detailed discussion of the existence and stability of dynamic equilibria in Diamond-models with production, see, in particular, Galor and Ryder (1989). For comprehensive surveys, see Azariadis (1993), von Thadden (1999) and de la Croix and Michel (2002), with the latter reference focusing also in detail on aspects of lump-sum tax and transfer systems, see p. 195 ff.
Invoking (A 4), we conclude this section, for further reference, with a brief discussion of why the two mentioned steady-state types are unstable. In general, according to the linearized system (9)-(10), dynamics without government debt dynamics are stable as long as $A_2 < A_1$. In the presence of government debt dynamics, however, instability can occur because of two partial effects. First, assuming a constant interest rate ($R'(k) = 0$), interest payments induce a snowball effect on debt, and this effect is unstable whenever the interest rate is higher than the (population) growth rate of the economy, i.e. whenever $R(k) > 1 + n$. Second, out of steady state the interest rate does not stay constant in an economy with capital stock dynamics, implying that, for any initial level of debt, there is an additional effect on debt according to $R'(k)b \cdot dk_t$. Any crowding out of capital leads over time to a higher interest rate which reinforces debt dynamics. We call this second channel the interest rate effect on debt.

**Benchmark 1: Underaccumulation steady state** $(k > 0, b = 0, R(k) > 1 + n)$

Since $b = 0$, the interest rate effect on debt in the linearized dynamics is zero and the instability is entirely caused by the snowball effect. Because of the absence of the interest rate effect, government debt dynamics are independent of (9) and it is easy to verify that the two eigenvalues of (9) and (10) are given by $\lambda_1 = A_2/A_1 \in (0, 1)$, and $\lambda_2 = R(k)/1 + n > 1$. This pattern of eigenvalues implies that, for initial values $k_0 \neq k$ and $b_0 \neq 0$ close to the steady state, dynamics are locally unstable under a balanced primary budget rule.

**Benchmark 2: Golden rule steady state** $(k > 0, b > 0, R(k) = 1 + n)$

At the golden rule steady state with positive debt, the snowball effect is associated with a unit root, and strict instability is ensured by the additionally operating interest rate effect on debt. This can be verified from the characteristic polynomial associated with (9) and (10), evaluated at the golden rule steady state:

$$p(\lambda) = \lambda^2 - [1 + \frac{A_2}{A_1} - \frac{R'(k)b}{A_1}] \cdot \lambda + \frac{A_2}{A_1}$$

Then, $p(0) = A_2/A_1 \in (0, 1)$ and $p(1) = R'(k)b/A_1 < 0$, implying $0 < \lambda_1 < 1 < \lambda_2$. Hence, at the golden rule steady state with $b > 0$ dynamics are locally unstable under a balanced primary budget rule.

### 3 Stability under a debt targeting rule: a common framework with three instruments

The common fiscal framework aims to ensure stability and is organized around a simple debt targeting rule which stipulates that the primary balance responds
to deviations of the two state variables $b_t$ and $k_t$ from their steady-state values according to

$$(1 + n) \cdot b_{t+1} = R(k_t)b_t - \pi(k_t - k, b_t - b),$$

where we pre-multiply $b_{t+1}$ with $1 + n$ in order to facilitate a simple comparison with the benchmark equation (5). Specifically, we assume that adjustments in the primary balance follow the linear feedback rule

$$\pi(k_t - k, b_t - b) = \pi(k, b) + \pi_k(k_t - k) + \pi_b(b_t - b) = \pi_t,$$

implying the generic debt targeting rule

$$(1 + n) \cdot b_{t+1} = R(k_t)b_t - \pi_k(k_t - k) - \pi_b(b_t - b).$$

(13)

Compared with the analysis of the previous section, we now allow for adjustments in the primary balance with the idea to stabilize the benchmark steady states, i.e. $\pi_t \neq 0$ is admitted for the off-steady-state dynamics. We distinguish between three scenarios in which adjustments are achieved by variations of one of the three available fiscal instruments $g_t$, $\eta_t$ or $\theta_{t+1}$, while holding the other two instruments constant at their steady-state values.

**Instrument 1: Variations in government spending $g_t$**

Assume the government satisfies (13) by adjusting government spending, according to $g_t - g = -\pi_t$. Then, the intertemporal equilibrium conditions can be summarized as

$$(1 + n)(k_{t+1} + b_{t+1}) = s(w(k_t) - \eta + \frac{\theta}{R(k_{t+1})}, R(k_{t+1})) - \frac{\theta}{R(k_{t+1})} \quad (14)$$

$$(1 + n) \cdot b_{t+1} = R(k_t)b_t - \pi_k(k_t - k) - \pi_b(b_t - b) \quad (15)$$

$$g_t = \tilde{g}(k_t, b_t) = g - \pi_k(k_t - k) - \pi_b(b_t - b) \quad (16)$$

Importantly, adjustments in the primary balance via variations in $g_t$ do not affect the accumulation equation, i.e. (14) is identical to (4). In other words, variations in $g_t$ offer a particularly convenient, non-distortionary channel to stabilize the two-dimensional benchmark system (7)-(8). Since (16) does not feed back into the first equation the linearized dynamics read as

$$A_1 \cdot dk_{t+1} + (1 + n) \cdot db_{t+1} = A_2 \cdot dk_t$$

(17)

$$(1 + n) \cdot db_{t+1} = (R'(k)b - \pi_k) \cdot dk_t + (R(k) - \pi_b) \cdot db_t$$

(18)

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13 Primary surpluses require $g_t < g$. Recall the assumption of $g > 0$. In the following, we assume that $g$ is sufficiently positive such that for the local dynamics around the steady state the inequality $g_t > 0$ is always satisfied.
Variations in the debt targeting rule which can be implemented without restricting the choice of the instrument.

**Instrument 2 : Variations in lump-sum taxes \( \eta_t \)**

We assume now, alternatively, that the government satisfies (13) by adjusting first-period lump-sum taxes such that \( \eta_t - \eta = \pi_t \). Then, the intertemporal equilibrium evolves according to

\[
(1 + n)(k_{t+1} + b_{t+1}) = s(w(k_t) - \tilde{\eta}(k_t, b_t) + \frac{\theta}{R(k_{t+1})}, R(k_{t+1})) - \frac{\theta}{R(k_{t+1})}
\]

\[
(1 + n)b_{t+1} = R(k_t)b_t - \pi_k(k_t - k) - \pi_b(b_t - b)
\]

\[
\eta_t = \tilde{\eta}(k_t, b_t) = \eta + \pi_k(k_t - k) + \pi_b(b_t - b)
\]

Again, dynamics are two-dimensional in \( k_t \) and \( b_t \) but adjustments in the primary balance via variations in \( \eta_t \) do affect the disposable income of young agents and, hence, the accumulation equation. Linearization of (19)-(20) yields

\[
A_1 \cdot dk_{t+1} + (1 + n) \cdot db_{t+1} = (A_2 - s_w \pi_k) \cdot dk_t - s_w \pi_b \cdot db_t
\]

\[
(1 + n) \cdot db_{t+1} = (R'(k)b - \pi_k) \cdot dk_t + (R(k) - \pi_b) \cdot db_t,
\]

where the link between the instrument and the accumulation equation is captured by the use of the partial derivatives \( \tilde{\eta}_k = \pi_k \) and \( \tilde{\eta}_b = \pi_b \).

**Instrument 3 : Variations in transfers \( \theta_{t+1} \)**

Finally, we consider the case where the government satisfies (13) by adjusting second-period transfers such that \( \theta_{t+1} - \theta = -(1 + n)\pi_{t+1} \). Then, the intertemporal equilibrium conditions are given by

\[
(1 + n)(k_{t+1} + b_{t+1}) = s(w(k_t) - \eta + \tilde{\theta}(k_{t+1}, b_{t+1}) + \frac{\theta}{R(k_{t+1})}, R(k_{t+1})) - \frac{\theta}{R(k_{t+1})}
\]

\[
(1 + n) \cdot b_{t+1} = R(k_t)b_t - \pi_k(k_t - k) - \pi_b(b_t - b)
\]

\[
\theta_{t+1} = \tilde{\theta}(k_{t+1}, b_{t+1}) = \theta - (1 + n) \cdot [\pi_k(k_{t+1} - k) + \pi_b(b_{t+1} - b)]
\]

Variations in \( \theta_{t+1} \) do affect the second-period disposable income of agents and, hence, the accumulation equation. This is reflected in the linearized versions of (23) and (24)

\[
[A_1 - (1 - s_w)\frac{(1 + n)}{R(k)} \pi_k] \cdot dk_{t+1} + [1 + n - (1 - s_w)\frac{(1 + n)}{R(k)} \pi_b] \cdot db_{t+1} = A_2 \cdot dk_t
\]

\[
(1 + n) \cdot db_{t+1} = (R'(k)b - \pi_k) \cdot dk_t + (R(k) - \pi_b) \cdot db_t,
\]

where we have used \( \tilde{\theta}_k = -(1 + n)\pi_k \) and \( \tilde{\theta}_b = -(1 + n)\pi_b \).

A common fiscal framework, in the context of this paper, is simply a representation of the debt targeting rule which can be implemented without restricting the choice
of instruments. For any steady state of (7) and (8) it is possible to derive three instrument-specific sets of feedback coefficients $\pi_k$ and $\pi_b$ which ensure locally stable dynamics. Depending on the particular steady state under consideration, however, it is not clear whether these three sets overlap, i.e. whether the debt targeting rule can be implemented under all instruments with a common set of feedback coefficients.

3.1 Underaccumulation steady state

This subsection shows that the first benchmark steady state derived above does have the particular feature that, without imposing further restrictions, all three instrument-specific sets of stabilizing feedback coefficients have a common intersection. This feature is directly linked to the absence of the interest rate effect on debt at this steady state, i.e. a deviation of the capital stock from its steady state value does not have by itself destabilizing effects on government debt dynamics. In other words, the instability of debt dynamics comes entirely from the snowball effect which can be fully corrected by an appropriate choice of $\pi_b$. Accordingly, for all three instruments, if the debt targeting rule is characterized by $\pi_k = 0$ government debt dynamics are independent of the accumulation equation and the two eigenvalues of the linearized dynamics are identically given by

$$\lambda_1 = \frac{A_2}{A_1} \in (0, 1), \quad \lambda_2 = \frac{R(k) - \pi_b}{1 + n}.$$  

Evidently, if $\pi_k = 0$ and $\pi_b \in (R(k) - (1 + n), R(k) + 1 + n)$ dynamics will be locally stable under all three instruments. Moreover, if one considers the subset characterized by $\pi_k = 0$ and $\pi_b \in (R(k) - (1 + n), R(k))$ dynamics will be locally stable and exhibit monotone adjustment for all three instruments. This reasoning leads to

**Proposition 1** Consider the three instrument-specific sets of feedback coefficients $\pi_k$ and $\pi_b$ which ensure local stability at the underaccumulation steady state under the debt targeting rule (13). These three sets have a joint intersection, i.e. there exist values for $\pi_k$ and $\pi_b$ such that the debt targeting rule achieves local stability under all three instruments.

**Corollary 1** There exist values for $\pi_k$ and $\pi_b$ such that the debt targeting rule achieves local stability and monotone adjustment to the steady state under all three instruments.

A more general illustration of Proposition 1 can be obtained if one recognizes that each of the three instruments is associated with a particular characteristic polynomial $p(\lambda)_i, i = g, \eta, \theta$ and that stability requires
\[ p(1)|_i > 0, \quad p(-1)|_i > 0, \quad p(0)|_i < 1, \quad \text{with: } i = g, \eta, \theta. \]

As derived in Appendix 1, for all three instruments these constraints (at equality) have a linear representation in \( \pi_b - \pi_k \)-space, giving rise to instrument-specific stability regions. From inspecting these constraints one can show

**Proposition 2** The three instrument-specific stability regions are not identical and for all three instruments there exist stabilizing feedback coefficients \( \pi_k \) and \( \pi_b \) which are not contained in the joint intersection.

For a proof: see Appendix 1.

Intuitively, Proposition 2 reflects that any feedback rule with \( \pi_k \neq 0 \) creates a policy-induced interdependence between capital stock dynamics and government debt dynamics. And this interdependence differs across the three instruments. For further illustration, consider

**Example 1:** \( F(K, L) = zK^\alpha L^{1-\alpha}, \quad U(c, d) = \ln c + \beta \ln d \)
\( \delta = 1, \quad z = 15, \quad \alpha = 0.4, \quad \beta = 0.5 \) (i.e. \( s_w = \beta/(1 + \beta) = 1/3 \)), \( \eta = -\theta = 2.91 \), \( g = 4.1, \quad 1 + n = 2.43 \). Assuming a period length of 30 years, this implies an annual population growth of 0.03. Assuming \( b = 0 \), one obtains \( R = 6.29 \) which corresponds to an annual real interest rate of 0.06, consistent with an underaccumulation steady state. Moreover, \( k = 0.92, \quad y = 14.53, \quad g/y = 0.28, \quad \eta/y = 0.2, \quad \theta/y = -0.2 \), i.e. agents have a similar steady-state tax burden in both periods, measured in terms of per capita output.

It is instructive to discuss first the stabilizing variations in government spending \( g_t \) because of their non-distortionary character. Corresponding to Example 1, Figure 1 plots the stability region in \( \pi_b - \pi_k \)-space for adjustments in \( g_t \). Points inside the triangle in bold line are associated with two stable eigenvalues. The first benchmark discussed in the previous section with coordinates \( \pi_b = \pi_k = 0 \), by construction, lies outside the stability triangle. The triangle reflects that there are potentially two margins for stabilizing adjustments of the primary balance, \( \pi_b \) and \( \pi_k \), which can be used in isolation or in combination. To further understand the shape of the triangle depicted in Figure 1 it is important to realize that there is one key difference between these two channels. Specifically, in the vicinity of any underaccumulation steady state with \( k_0 \neq k \) and \( b_0 \neq 0 \), only debt imbalances, because of the snowball effect, destabilize on impact government debt dynamics, and consolidations according to \( \pi_b \) react immediately to this instability. By contrast, consolidations according to \( \pi_k \) respond with the delay of one period to the snowball effect and only to the extent that it leads to the crowding out of capital. Because of the different timing
of the reactions under the two channels, stabilization can always be achieved if the primary surplus exclusively reacts to the debt imbalance, i.e. if \( \pi_b = 0 \) local stability is ensured if \( \pi_b \in (R(k) - (1 + n), R(k) + 1 + n) \). By contrast, if the primary surplus exclusively reacts to the capital stock imbalance stabilization may not be possible, i.e. if \( \pi_b = 0 \) there does not necessarily exist a range for \( \pi_k \) such that local stability is ensured, as illustrated in Figure 1.\(^{14}\) Hence, the effectiveness of the two fiscal feedback margins, when considered in isolation, is different, reflecting the general principle that, from a stabilization perspective, imbalances should be addressed directly at their source rather than indirectly and with some delay.

In any case, there exists a wide range of combinations of the two feedbacks which are consistent with locally stable dynamics. In particular, assume that \( \pi_b < R(k) - (1 + n) \), i.e. there is no fully stabilizing direct reaction via \( \pi_b \) to the snowball effect. Then, if \( \pi_k \) is sufficiently negative (i.e. there is a sufficient reaction to the crowding out of capital) overall dynamics may nevertheless be stable. Conversely, assume that \( \pi_b > R(k) + (1 + n) \), i.e. the direct reaction to debt imbalances overshoots and risks destabilizing fluctuations. Then, it may nevertheless be possible to have overall stable dynamics if \( \pi_k \) is sufficiently positive.

In general, stable pairs of feedback coefficients inside the stability triangle of Figure 1 are associated with a wide range of possible adjustment patterns. In particular, the area to the southwest of the hyperbola is associated with endogenously possible adjustments. In particular, assume that \( \pi_b < R(k) - (1 + n) \), i.e. there is no fully stabilizing direct reaction via \( \pi_b \) to the snowball effect. Then, if \( \pi_k \) is sufficiently negative (i.e. there is a sufficient reaction to the crowding out of capital) overall dynamics may nevertheless be stable. Conversely, assume that \( \pi_b > R(k) + (1 + n) \), i.e. the direct reaction to debt imbalances overshoots and risks destabilizing fluctuations. Then, it may nevertheless be possible to have overall stable dynamics if \( \pi_k \) is sufficiently positive.

In general, stable pairs of feedback coefficients inside the stability triangle of Figure 1 are associated with a wide range of possible adjustment patterns. In particular, the area to the southwest of the hyperbola is associated with endogenously fluctuating adjustment dynamics because of complex eigenvalues. Note that only points within the small, shaded triangle \( ABD \) are associated with two stable and positive eigenvalues, ensuring monotone adjustment dynamics. Moreover, Figure 1 indicates that the combination of \( \pi_b \) and \( \pi_k \) associated with the highest speed of adjustment \( (\lambda_1 = \lambda_2 = 0) \), denoted by the point \( B \), requires reactions along both margins. One easily verifies in (17) and (18) that the point \( B \) has coordinates \( \pi_b = R(k) \) and \( \pi_k = -A_2 \). Intuitively, to fully offset any initial deviation from the steady-state values, the response of the primary balance should not only neutralize the debt imbalance \( (\pi_b = R(k)) \), but also fully correct for the disturbed savings behaviour of young agents \( (\pi_k = -A_2) \). More generally, the location of \( B \) indicates that for any given stabilizing direct reaction to debt imbalances, measured in terms of \( \pi_b \), variations in \( \pi_k \) lead to different speeds of adjustments of the system. This is illustrated in Figure 2 which plots the impulse response of the system to an initial constellation \( b_0 > 0 \) and \( k_0 < k \) for \( \pi_b = R(k) \) and three distinct values of \( \pi_k : i) \pi_k = -A_2 \)

\(^{14}\)However, this finding depends on the strength of the snowball effect. To further illustrate this, consider example 1 and assume, everything else being equal, \( \alpha = 1/3 \). Because of the higher wage income share this induces, ceteris paribus, higher savings and at \( b = 0 \) a lower return factor \( R = 4.35 \), implying an annual real interest rate of 0.05, i.e. the snowball effect will be smaller than in example 1. Then, \( R - A_1/A_2(1 + n) < 0 \), and full stabilization can be achieved if \( \pi_b = 0 \) and if \( \pi_k \) takes on appropriate negative values. Apart from the leftward shift of the stability triangle, however, the features of Figure 1 remain qualitatively unchanged.
(maximum speed of adjustment at point $B$, implying $\lambda_1 = \lambda_2 = 0$); ii) $\pi_k = 0$ (intermediate speed of adjustment at point $C$ which, by construction, corresponds in terms of $k_t$ to the Diamond model without debt dynamics and has $\lambda_1 = 0$ and $\lambda_2 = A_2/A_1 \in (0, 1)$; iii) $\pi_k = A_1 - A_2 - \varepsilon$ (slow speed of adjustment at a point close to $D$ with $\lambda_1 = 0$ and $\lambda_2 = 1 - \varepsilon/A_1$, i.e. by choosing some small $\varepsilon > 0$ the second root can be made arbitrarily close to unity and Figure 2 uses for illustration $\varepsilon = 0.1$).

Finally, Figure 3 also includes the stability regions for the other two instruments which distort the accumulation equation. It is worth pointing out that for small values of $\pi_b$ the area corresponding to adjustments in $\eta_t$ leaves less scope for compensating reactions of the primary balance to capital stock imbalances. Intuitively, this is the case since the disposable income of young agents depends negatively on first-period tax payments $\eta_t$. Hence, if the fiscal rule attempts to stabilize the unstable snowball effect, at least partly, via responses to capital stock imbalances this introduces not only a costly delay, but it also diminishes the disposable income of young agents and, hence, savings which are needed in the first place to support higher investments. Because of this effect, there is less scope to substitute delayed reactions via $\pi_k$ for direct reactions via $\pi_b$ than under non-distortionary variations of $g_t$.

By contrast, if the fiscal adjustment is instead achieved via reduced second-period transfers $\theta_{t+1}$ the same mechanism works in the opposite direction since this encourages savings. Consequently, for small values of $\pi_b$ the stability region associated with variations in $\theta_{t+1}$, is much wider than the one associated with both $g_t$ and $\eta_t$. More specifically, as one infers from Figure 3, the stability region associated with variations in $\theta_{t+1}$ is not bounded from below. This reflects that under this regime debt-stabilizing fiscal measures lead to higher savings, reinforcing thereby the overall stability of the system.

### 3.2 Golden rule steady state

As discussed in Section 2, at the golden rule steady state with $b > 0$ the interest rate effect on debt is a key margin of instability, giving rise to two distinct features. First, it is impossible to address this instability directly by means of adjustments via $\pi_k$ and, at the same time, to obtain a recursive dynamic structure which insulates the accumulation equation of the Diamond-model under all three instruments against the stabilization of debt dynamics. Second, to address this instability indirectly through adjustments via $\pi_b$ comes with a delay and, depending on the strength of the interest rate effect on debt, this delay can be very costly in terms of destabilizing dynamics. In combination, these two features give rise to the strong result:
Proposition 3  Consider the three instrument-specific sets of feedback coefficients $\pi_k$ and $\pi_b$ which ensure under the debt targeting rule (13) local stability at the golden rule steady state. These three sets do not always have a joint intersection, i.e. it is possible that there exist no values for $\pi_k$ and $\pi_b$ such that the debt targeting rule achieves local stability under all three instruments.

Proof: See Appendix 1

To explain intuitively the logic underlying Proposition 3, we proceed in two steps. First, extending Example 1, we choose a parametrization which leads to a golden rule steady state with a ‘small’ debt ratio:

Example 2: Consider example 1, but let $\alpha = 0.2$, $\eta = -\theta = 3.16$, $g = 4.46$. Assuming $b = 0$, one obtains $R = 1.76 < 1 + n = 2.43$. At the golden rule steady state, $R = 1 + n = 2.43$, yielding an annual real interest rate of 0.03, $b = 0.36 > 0$, $k = 1.3$, $y = 15.8$ and a debt ratio of $b/y = 0.02$. Moreover, $g/y = 0.28$, $\eta/y = 0.2$, $\theta/y = -0.2$, i.e. agents have a steady-state tax burden in both periods as in example 1.

Figure 4 illustrates the stability regions for variations in all three instruments. By construction, the second benchmark discussed in Section 2 with coordinates $\pi_b = \pi_k = 0$ lies outside all three instrument-specific stability regions. Figure 4 shares with Figure 3 the feature that the three regions have a common intersection. Nevertheless, as far as the region associated with variations in $g_t$ is concerned, there is one particularly important difference. Specifically, since the interest rate effect on debt is now a key margin of instability, stabilization can always be achieved if the primary surplus exclusively reacts to the capital stock imbalance, i.e. if $\pi_b = 0$ local stability is always ensured for some values $\pi_k < 0$. If the primary surplus exclusively reacts to the debt imbalance (i.e. if $\pi_k = 0$) stabilization can be achieved for this particular parametrization. However, the latter finding holds not always true. To illustrate this, we look at

Example 3: Consider example 1, but let now $\alpha = 0.15$ and $\beta = 1$ (i.e. $s_w = \beta/(1 + \beta) = 1/2$). Moreover, $\eta = 0.74$, $\theta = -6.63$, $g = 3.47$. Assuming $b = 0$, one obtains $R = 0.32 < 1 + n = 2.43$. At the golden rule steady state, $R = 1 + n = 2.43$, yielding $b = 2.11 > 0$, $k = 0.89$, $y = 14.74$ and a debt ratio of $b/y = 0.14$. Moreover, $g/y = 0.24$, $\eta/y = 0.05$, $\theta/y = -0.45$, i.e. the tax burden of agents in the second period is now substantially higher than in the first.

Example 3 leads overall to higher savings of young agents by assuming a higher wage income share, a higher propensity to save out of disposable income and by shifting taxes from the youth to the old age of agents. In sum, because of stronger
accumulation dynamics, this leads to a golden rule debt ratio which is substantially higher than under example 2. As illustrated in Figure 5, this gives rise to a constellation where under the $g_t$-regime the stabilization of debt dynamics through exclusive reactions to debt imbalances (i.e. $\pi_k = 0$) is not possible. Moreover, by considering also the stability region for variations in $\eta_t$, this example illustrates Proposition 3, as also shown in Figure 5. Generally speaking, the two stability regions associated with variations in $g_t$ and $\eta_t$ have no longer a common intersection, since the high interest rate effect on debt requires a stabilizing response via $\pi_k$ which leads to rather different accumulation equations under the $g_t$-regime and the $\eta_t$-regime. For an intuitive explanation, it is important to note that in Figure 5 the stability triangle associated with the $\eta_t$-regime is to the southeast of the stability triangle associated with the $g_t$-regime. Moreover, under the $g_t$-regime points to the southeast of the $g_t$-triangle have one unstable eigenvalue. For the sake of the argument consider an initial constellation with $b_0 > b$ and $k_0 < k$. Then, under the $g_t$-regime for feedback coefficients to the southeast of the $g_t$-triangle there is, for given savings, too much stabilization of debt dynamics, i.e. there is too little emission of new bonds $b_{t+1}$. This implies that the composition of next period’s assets $(k_{t+1} + b_{t+1})$ becomes too productive, relative to the capacity of the economy to absorb investments in capital. However, points to the southeast of the $g_t$-triangle may nevertheless be consistent with fully stabilizing dynamics under the $\eta_t$-regime. The reason for this is that under the $\eta_t$-regime total savings will be lower because of the tax burden imposed on young agents. Because of this there is less scope that a strong reduction of $b_{t+1}$ can trigger ‘overinvestment’ in physical capital $k_{t+1}$. In sum, this reasoning shows that the instrument-specific reactions to imbalances may not only be different, but also mutually exclusive if one wishes to maintain over time the knife-edge portfolio composition between government bonds and physical capital at the golden rule steady state.

4 Extensions

4.1 Alternative representations of the debt targeting rule

It is worth pointing out that there exist alternative representations of the debt targeting rule which lead to the same results summarized in Propositions 1 – 3. We consider two particularly intuitive alternatives. As a starting point, we repeat the budget constraint of the government

$$(1 + n) \cdot b_{t+1} = R(k_t) \cdot b_t - \pi_t,$$

and maintain the assumption that at the steady states under consideration the primary balance is zero, i.e. $\pi_t = 0$. 

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First, let us assume that the debt targeting rule is now expressed in terms of stabilizing reactions of the overall deficit $\Delta_t$, using

\[
(1 + n) \cdot b_{t+1} = b_t + \Delta_t, \\
\Delta_t = \Delta(k_t, b_t) = (R(k_t) - 1) \cdot b_t - \pi_t
\]

Note that the deficit, at any moment in time, consists of a predetermined component linked to interest payments on debt, and a policy component linked to the primary balance. Only the latter part can actively react to the two predetermined states of the economy, $b_t$ and $k_t$. Accordingly, a debt targeting rule with stabilizing reactions of the deficit to the states of the economy, in linearized form, needs to be established from

\[
(1 + n) \cdot db_{t+1} = \Delta_k \cdot dk_t + (1 + \Delta_b) \cdot db_t, \\
\text{with} \quad \Delta_k = R'(k) \cdot b - \pi_k, \quad \text{and} \quad \Delta_b = R(k) - 1 - \pi_b.
\]

Consider the three linearized dynamic systems (17)-(18), (21)-(22), and (25)-(26) which were derived above for the three instruments. Using (27) and replacing $\pi_k$ and $\pi_b$ by the terms $R'(k) \cdot b - \Delta_k$ and $R(k) - 1 - \Delta_b$, respectively, these three systems can be transformed into three new systems, all exhibiting two-dimensional dynamics in $k_t$ and $b_t$. However, since $R(k)$, $R'(k)$, and $\pi$ are all predetermined the switch from the representation in $\pi_b - \pi_k$-space to a representation in $\Delta_b - \Delta_k$-space amounts simply to an affine transformation, leaving the results of Propositions 1–3 unaffected.

Second, the debt targeting rule can be reinterpreted as a rule which expresses the issuance of new per capita debt directly in terms of stabilizing reactions to the states of the economy, according to

\[
(1 + n) \cdot b_{t+1} = h_t, \\
h_t = h(k_t, b_t) = R(k_t) \cdot b_t - \pi_t.
\]

After linearizing (29), it is clear that a switch from a representation in $\pi_b - \pi_k$-space to a representation in $h_b - h_k$-space amounts to another affine transformation, leaving, again, the results of Propositions 1–3 unaffected.

### 4.2 Endogenous labour supply and distortionary taxes

The purpose of this subsection is to show that all the central findings of Section 3, as summarized by Propositions 1–3, prevail qualitatively in a richer setting which is characterized by an endogenous labour supply and distortionary taxes. Yet, there is an interesting twist to the results of the richer setting which is worth pointing
out. To this end, we assume now that preferences are described by the more general expression

\[ U(c_t - \phi(l_t), d_{t+1}), \]

where \( l_t \) denotes the variable labour supply of the representative young agent and the function \( \phi(l_t) \) captures the disutility of work, with \( \phi(0) > 0 \), and \( \phi'(l_t) > 0 \), \( \phi''(l_t) > 0 \) for all \( l_t > 0 \). As discussed in Greenwood, Hercowitz, and Huffman (1988), this labour supply specification has the convenient feature that it can be solved independently from the intertemporal consumption and savings decisions, allowing for easy comparability with the analysis of the previous section. Moreover, also for simple comparability, it is assumed that total tax revenues have a lump-sum and a distortionary component

\[ \eta_t = \eta + \tau_t w_t l_t, \quad (30) \]

where \( \tau_t \) denotes the wage income tax rate. We maintain the assumption that for the steady-state tax-transfer system, characterized by \( g = \eta - \frac{\theta}{1 + \nu} \), only lump-sum taxes matter. In contrast to the previous analysis, however, the entire out-of-steady-state adjustment burden falls on variations of the distortionary component \( \tau_t \), whenever tax adjustments are the preferred instrument to stabilize the debt level around the steady-state target. Accordingly, the objective can be replaced by

\[ U(w_t l_t - \phi(l_t) - \eta_t - s_t, R_{t+1} s_t + \theta_{t+1}), \]

giving rise to the pair of first order conditions

\[ \phi'(l_t) = (1 - \tau_t)w_t \]

\[ U_1 = R_{t+1} U_2. \quad (32) \]

With a flexible labour supply, the labour market equilibrium condition becomes \( L_t = l_t N_t \) and the first-order conditions from the profit maximization of firms are given by

\[ R_t = 1 - \delta + F_K(k_t, l_t) \]

\[ w_t = F_L(k_t, l_t). \quad (34) \]

Combining (30), (31), and (34) yields

\[ l_t \phi'(l_t) = l_t F_L(k_t, l_t) - (\eta_t - \eta), \]

which implicitly defines the equilibrium labour supply

\[ l_t = l(k_t, \eta_t) \]
in the vicinity of some steady state \( l = l(k, \eta) \), with partial derivatives \( l_k(k, \eta) = \frac{\partial F_{2,K}(k, \eta)}{\partial k} > 0 \) and \( l_\eta(k, \eta) = \frac{\partial F_{2,K}(k, \eta)}{\partial \eta} < 0 \). To keep the structure of the analysis as similar as possible to Section 2, we define the adjusted gross wage income net of the disutility term \( \varphi(l_t) \) as

\[
\tilde{w}_t = w_t - \varphi(l_t) = F_L(k_t, l_t) \cdot l_t - \varphi(l_t) = \tilde{w}(k_t, \eta_t). \tag{35}
\]

Using (35), the savings function reduces to

\[
s_t = s(\tilde{w}_t - \eta_t + \frac{\theta_{t+1}}{R_{t+1}}, R_{t+1}) - \frac{\theta_{t+1}}{R_{t+1}}.
\]

Moreover, key features of the factor price frontier remain qualitatively unchanged compared with Section 2 if one expresses the factor prices in terms of the ‘adjusted return rates’

\[
R_t = 1 - \delta + F_K(k_t, l_t) = R(k_t, \eta_t)
\]

\[
\tilde{w}_t = \tilde{w}(k_t, \eta_t).
\]

In particular, \( R_t \) falls in \( k_t \), and \( \tilde{w}_t \) rises in \( l_t \), since

\[
R_k = F_{KK} + F_{KL} \cdot l_k = F_{KK} \cdot \frac{\varphi''}{\varphi'' - F_{LL}} < 0 \tag{36}
\]

\[
\tilde{w}_k = l \cdot (F_{LK} + F_{LL} \cdot l_k) = l \cdot F_{LK} \cdot \frac{\varphi''}{\varphi'' - F_{LL}} > 0. \tag{37}
\]

Similarly, one obtains

\[
R_\eta = F_{KL} \cdot l_\eta < 0 \tag{38}
\]

\[
\tilde{w}_\eta = l \cdot F_{LL} \cdot l_\eta > 0, \tag{39}
\]

i.e. the adjusted return rates move upon a change in the distortionary labour tax in different directions, reflecting that the labour supply itself falls in \( \eta \). To keep the structure of the partial derivatives use the notation \( l_k(k, \eta) = \frac{\delta l_k}{\delta k} \bigg|_{k_t=k} \) and \( l_\eta(k, \eta) = \frac{\delta l_\eta}{\delta \eta} \bigg|_{\eta_t=\eta} \), i.e. in the latter case the derivative is taken with respect to \( \eta_t \) and then evaluated at the steady state characterized by \( \eta_t = \eta \).

To establish (36) we exploit \( F_{KK} \cdot F_{LL} = (F_{KL})^2 \) which follows from the linear homogeneity assumption made in (A 4).

Note that \( R_\eta < 0 \) implies that the gross wage rate \( w = F_L(k, l) \) rises in \( \eta \), i.e. \( w_\eta = F_{LL} \cdot l_\eta > 0 \). To see why the adjusted wage term \( \tilde{w} \) satisfies \( \tilde{w}_\eta > 0 \) note first that the gross wage bill \((wl)\) falls in \( \eta \), despite the rise in \( w \), if the wage elasticity of employment is larger than one. However, the disutility of labour decreases because of the reduced labour supply, and the latter effect always dominates, ensuring \( \tilde{w}_\eta > 0 \). Finally, it is worth pointing out that the net wage rate \((1 - \tau_\eta)w = w - (\eta_t - \eta)/l \), evaluated at the steady state, falls in \( \eta \), since \( w_\eta - 1/l < 0 \) if \(-F_{LL}/[\varphi''(l) - F_{LL}] < 1 \), and the latter inequality must always be satisfied.
set of intertemporal equilibrium conditions can be summarized as

\[(1 + n)(k_{t+1} + b_{t+1}) = s(\tilde{w}(k_t, \eta_t) - \eta_t + \frac{\theta_{t+1}}{R(k_{t+1}, \eta_{t+1})}, R(k_{t+1}, \eta_{t+1})) - \frac{\theta_{t+1}}{R(k_{t+1}, \eta_{t+1})}\]  

\[(1 + n)b_{t+1} = R(k_t, \eta_t)b_t - \pi_t \]  

\[\pi_t = \eta_t - \frac{\theta_t}{1 + n - g_t} \]  

The system (40)-(42) is structurally similar to the system (4)-(6) discussed in the previous section. Specifically, let

\[A^*_1 = 1 + n - R_k \cdot [s_R + (1 - s_w) \frac{\theta}{R(k)^2}] \]

\[A^*_2 = s_w \tilde{w}_k > 0 \]

and assume \(A^*_1 > A^*_2\). Then, in the light of (36) and (37), dynamics remain qualitatively unchanged for variations in \(g_t\) and \(\theta_{t+1}\), since \(\eta_t\) will be held constant in these scenarios. If, however, adjustments in the primary balance are achieved via distortionary wage income taxes, as embodied in \(\eta_t\), the dynamic system behaves qualitatively differently from Section 3 because of the additional partial effects resulting from (38) and (39).\(^{18}\) The key difference is that variations in the wage income tax affect the labour supply and, hence, the (pre-tax) factor return rates. As indicated by (38) and (39), the reduced labour supply decreases the equilibrium interest factor \(R_t\) and increases both the wage rate \(w_t\) and the adjusted wage term \(\tilde{w}_t\). This feature implies that the distortionary wage income tax acts in terms of factor prices like a built-in-stabilizer which moderates the destabilizing interest rate effect on debt. In other words, whenever wage income taxes are changed to address unstable debt dynamics this has the convenient implication that the interest rate effect on debt will be endogenously dampened through the mechanics of the factor-price frontier, assuming competitive factor markets and an elastic labour supply. As we show in the final two sections, this feature does not affect the assessment of underaccumulation steady states, but it somewhat moderates the assessment of golden rule steady states.

### 4.2.1 Underaccumulation steady state

At any underaccumulation steady state the interest rate effect on debt is zero and the moderation of factor prices under the \(\eta_t\)-regime is therefore, qualitatively, without

\(^{18}\)The following analysis, alternatively, could have been carried out in terms of variations in the direct tax instrument \(\tau_t\), while the here considered variations in \(\eta_t\) take implicitly also the reaction of the labour supply to changes in \(\tau_t\) into account. With \(\tau_t\) and \(\eta_t\) evaluated at their respective steady state values, one can show that the linearized dynamic systems are identical for the two approaches. Hence, none of the results established in this section depend on this modelling choice.
consequence for the structure of the dynamic system. Recall from Section 3.1 that
\[ \eta_t = \eta_t(k_t, b_t) = \eta + \pi_k(k_t - k) + \pi_b(b_t - b), \]
with \( \tilde{\eta}_k = \pi_k \) and \( \tilde{\eta}_b = \pi_b \). Appendix 2 summarizes for all three instruments key features of the linearized dynamic equations. Because of the structural similarities between the systems (40)-(42) and (4)-(6) the main result of this section, however, can be entirely inferred from the linearized version of (41):
\[ (1 + n) \cdot db_{t+1} = [(R_k + R_{\eta} \pi_k)b - \pi_k] \cdot dk_t + [R(k) - \pi_b + R_{\eta} \pi_b b] \cdot db_t. \tag{43} \]
Since \( b = 0 \), (43) turns under the particular assumption of \( \pi_k = 0 \) into the one-dimensional dynamic equation in \( b_t \) and \( b_{t+1} \):
\[ db_{t+1} = \frac{R(k) - \pi_b}{1 + n} \cdot db_t. \]
Hence, for all three instruments \( g_t, \eta_t \), and \( \theta_{t+1} \) the two eigenvalues of the system are identically given by
\[ \lambda_1 = A^*_{22}/A^*_{11} \in (0, 1), \quad \lambda_2 = \frac{R(k) - \pi_b}{1 + n}. \]
if one assumes \( \pi_k = 0 \). Because of this feature, the classification of dynamic equilibria under the three instruments is as in Section 3.1. and the Propositions 1 and 2 remain unaffected.

4.2.2 Golden rule steady state

At any golden rule steady state, Proposition 3 of Section 3.2 remains valid, but the stabilizing reactions of factor prices under the \( \eta_t \)-regime make it in a certain sense less likely that the instrument-specific sets of stable feedback coefficients \( \pi_k \) and \( \pi_b \) have no joint intersection. To operationalize this insight, it is convenient to reconsider the example economy used so far with a more general preference structure. Specifically, let
\[ U(c_t - \varphi(l_t), d_{t+1}) = \ln[c_t - \frac{\xi}{1 + \chi} l_t^{1+\chi}] + \beta \ln d_{t+1}, \tag{44} \]
where \( \chi > 0 \) denotes the inverse of the constant elasticity of the labour supply. As shown in Appendix 2, when combined with a Cobb-Douglas production function, (44) implies for the two crucial partial effects on return rates (38) and (39):
\[ R_{\eta} = F_{KL} \cdot l_{\eta} = -\frac{\alpha}{\alpha + \chi} \cdot \frac{1}{k} < 0 \tag{45} \]
\[ \tilde{w}_{\eta} = l \cdot F_{LL} \cdot l_{\eta} = \frac{\alpha}{\alpha + \chi} \in (0, 1), \tag{46} \]

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where $\alpha \in (0, 1)$ denotes the Cobb-Douglas share of capital. Evidently, the parameter $\chi$ is of key importance for the reactions of the factor prices to changes in $\eta$. Specifically, as $\chi$ becomes large the labour supply becomes inelastic and the economy behaves qualitatively like the benchmark scenario discussed in Section 3, since $R_\eta \to 0$ and $\bar{w}_\eta \to 0$. For illustration, example 4 sets $\chi = 1000$ and uses numerical values for the other parameters which reproduce example 3, i.e. the $g_t$-regime and the $\eta_t$-regime have no common intersection in terms of stabilizing feedback coefficients. By contrast, example 5 drops the assumption of an inelastic labour supply and uses instead a much lower value of $\chi = 2$.

Example 4: $F(K, L) = z K^\alpha L^{1-\alpha}$, $U(c, l, d, \ldots) = \ln[c - \frac{\xi}{1+\chi}] + \beta \ln d$.

$\chi = 1000, \delta = 1, z = 15, \alpha = 0.15, \beta = 1, R = 1 + n = 2.43$. Moreover, $\eta = 0.74, \theta = -6.63, g = 3.46$, implying $b = 2.11 > 0, k = 0.88, y = 14.72, b/y = 0.14, g/y = 0.24, \eta/y = 0.05, \theta/y = -0.45$. The ‘free’ parameter $\xi$ is set at $\xi = 30$, normalizing the labour supply to $l = 1$.

Example 5: Consider example 4, but let $\chi = 2$. We maintain $g/y = 0.24, \eta/y = 0.05, \theta/y = -0.45$. Everything else being equal, this implies for the remaining endogenous variables: $R = 1 + n = 2.43, b = 0.81 > 0, k = 0.57, y = 9.54, b/y = 0.08, l = 0.6$.

Figures 6 and 7 show for the two example economies the stability regions associated with all three instruments. The key finding is that under the elastic labour supply of example 5 the stability triangle of the $\eta_t$-regime is no longer strictly to the southeast of the $g_t$-triangle, but allows instead for a common intersection. In other words, Figures 6 and 7 illustrate that under distortionary wage income taxes there is some scope for stabilizing reactions of factor prices which tends to moderate the strong result of Proposition 3.

5 Conclusion

This paper studies the stabilization of government debt dynamics under a number of different fiscal instruments from a comparative perspective. Specifically, the paper addresses the question of whether a state contingent debt targeting rule which links the stabilization of debt to the underlying state of the economy can be implemented under all available fiscal instruments with a common set of feedback coefficients. The main result of the paper is that the answer to this question cannot be given without reference to the level of long-run debt around which the economy is stabilized. Intuitively, this finding reflects that different fiscal instruments (like government spending, public transfers, and the menu of available taxes) affect the economy through different, instrument-specific margins. The level of debt to be
stabilized determines the weight of these margins within the set of intertemporal conditions. Hence, as the level of debt rises the importance of these margins increases and eventually the instrument-specific adjustment paths become sufficiently diverse such that it is no longer possible to implement the debt targeting rule within a framework of common feedback coefficients.

As the paper stands, these results are derived in a deliberately small and fully tractable model of a closed economy. Yet, the policy implications can probably best be seen in the context of a monetary union with decentralized fiscal policies, subject to certain provisions of a common fiscal framework of surveillance. The analysis of this paper does not add any new arguments why such a framework is necessary. Instead, it indicates that within any such framework high levels of average debt are likely to create tensions between the necessary provisions of a common framework and the unrestricted choice of fiscal instruments at the national level. This paper implicitly assumes that the latter feature is by itself of considerable value. To preserve this value under the conditions of a monetary union, however, essentially implies from the perspective of this paper that the union’s fiscal framework should be organized around a sufficiently ambitious target level of debt. We leave it for future work to further explore this mechanism, also with a focus on quantitative issues, within a modelling framework that explicitly allows for a set-up with multiple countries.
References


Appendix 1: Fixed labour supply and lump sum taxes

Preliminaries to the proofs of Proposition 2 and 3:

For further reference, we derive for the three instruments the characteristic polynomials $p(\lambda)_i$ and the critical stability conditions $p(1)|_i = 0$, $p(-1)|_i = 0$, $p(0)|_i = 1$, for: $i = g, \eta, \theta$.

1) Regarding $g$, the characteristic equation associated with (17) and (18) is given by

$$p(\lambda)|_g = \lambda^2 - \left[\frac{A_2 - A_1\lambda}{A_1} + \frac{R(k) - \pi_b}{1 + n} - \frac{R'(k)b - \pi_k}{A_1}\right] \cdot \lambda + \frac{A_2}{A_1(1+n)}(R(k) - \pi_b)$$

$$p(0)|_g = \frac{A_2 - A_1\lambda}{A_1} + \frac{R(k) - \pi_b}{1 + n} - \frac{R'(k)b - \pi_k}{A_1}$$

$$p(1)|_g = (1 - \frac{A_2}{A_1})(1 - \frac{R(k) - \pi_b}{1 + n}) + \frac{R'(k)b - \pi_k}{A_1}$$

$$p(-1)|_g = (1 + \frac{A_2}{A_1})(1 + \frac{R(k) - \pi_b}{1 + n}) - \frac{R'(k)b - \pi_k}{A_1}$$

2) Regarding $\eta$, the characteristic equation associated with (21) and (22) obeys

$$p(\lambda)|_\eta = p(\lambda)|_g + \frac{s_w\pi_k}{A_1} \cdot \lambda + s_w \cdot \frac{R'(k)b\pi_b - R(k)\pi_k}{A_1(1+n)}$$

$$p(0)|_\eta = p(0)|_g + s_w \cdot \frac{R'(k)b\pi_b - R(k)\pi_k}{A_1(1+n)}$$

$$p(1)|_\eta = p(1)|_g + s_w \cdot \frac{R'(k)b\pi_b + [(1 + n) - R(k)]\pi_k}{A_1(1+n)}$$

$$p(-1)|_\eta = p(-1)|_g + s_w \cdot \frac{R'(k)b\pi_b - [(1 + n) + R(k)]\pi_k}{A_1(1+n)}$$

3) Regarding $\theta_{t+1}$, the characteristic equation associated with (25) and (26) results from

$$p(\lambda)|_{\theta_{t+1}} = p(\lambda)|_g + \frac{(1-s_w)(4+\theta)\pi_k}{R} \cdot \lambda - [1 + n - (1-s_w)(4+\theta)\pi_k] \cdot \frac{R'(k)b - \pi_k}{R(k) - \pi_b - (1 + n)\lambda}$$
Let $\tilde{A}_1 = A_1 - (1 - s_w)\frac{1+n}{R(k)}\pi_k$. Note that $\tilde{A}_1 = A_1$ if $\pi_k = 0$. Then:

$p(\lambda)_{|p} = \lambda^2 - \theta_1 \lambda + \theta_2$, where

$p(0)_{|p} = \frac{A_2}{A_1}R(k) - \frac{R'(k)b - \pi_k}{A_1} + (1 - s_w)\frac{1+n}{R(k)}\pi_b(R'(k)b - \pi_k)\frac{A_1}{A_1(1+n)}$

$p(0)_{|p} = \frac{A_2}{A_1}R(k) - \frac{A_2}{A_1(1+n)}\pi_b$

$p(1)_{|p} = (1 - \frac{A_2}{A_1})(1 - \frac{R(k) - \pi_b}{A_1}) + \frac{R'(k)b - \pi_k}{A_1} - (1 - s_w)\frac{\pi_b}{A_1R(k)}(R'(k)b - \pi_k)$

$p(-1)_{|p} = (1 + \frac{A_2}{A_1})(1 + \frac{R(k) - \pi_b}{A_1}) - A_2\frac{R(k)}{A_1(1+n)} + (1 - s_w)\frac{\pi_b}{A_1R(k)}(R'(k)b - \pi_k)$

**Proof of proposition 2:**
Consider Figure 3. Then, independent of the particular functional forms underlying example 1, the stability constraints at the underaccumulation steady state with $R(k) > 1 + n$ and $b = 0$ satisfy:

i) If $\pi_k = 0$, then $p(0)_{|p} = p(0)_{|\eta} = p(0)_{|\theta} = 1$ jointly intersect at $\pi_{b,0} = R(k) - (1 + n)A_1/A_2$. If $\pi_k = 0$, then $p(1)_{|p} = p(1)_{|\eta} = p(1)_{|\theta} = 0$ jointly intersect at $\pi_{b,1} = R(k) - (1 + n) > \pi_{b,0}$. If $\pi_k = 0$, then $p(-1)_{|p} = p(-1)_{|\eta} = p(-1)_{|\theta} = 0$ jointly intersect at $\pi_{b,-1} = R(k) + (1 + n) > \pi_{b,1}$.

ii) $p(0)_{|\eta} = 1$ is vertical in $\pi_b - \pi_k$-space. Moreover, $p(0)_{|\eta} = 1$ slopes downward and $p(0)_{|\theta} = 1$ slopes upward, since

$p(0)_{|\eta} = 1 \Leftrightarrow \pi_k = A_2R(k) - A_1(1+n) - \frac{A_2}{s_wR(k)}\pi_b$

$p(0)_{|p} = 1 \Leftrightarrow \pi_k = \frac{R(k)}{(1-s_w)(1+n)}(A_1 - A_2)\frac{R(k)}{1+n} + \frac{A_2}{1-s_w(1+n)}\frac{R(k)}{1+n}\pi_b$

iii) $p(1)_{|\eta} = 0$, $p(1)_{|\theta} = 0$ all slope upward in $\pi_b - \pi_k$-space, with

$p(1)_{|\eta} = 0 \Leftrightarrow \pi_k = (A_1 - A_2)(1 - \frac{R(k)}{1+n}) + \frac{A_1 - A_2}{1+n}\pi_b$

$p(1)_{|\eta} = 0 \Leftrightarrow \pi_k = (\frac{A_1 - A_2}{1-s_w(1+n)} + s_wR(k)) + \frac{A_1 - A_2}{1-s_w(1+n)}\frac{R(k)}{1+n}\pi_b$

$p(1)_{|\theta} = 0 \Leftrightarrow \pi_k = (\frac{A_1 - A_2}{1-s_w(1+n)} + s_wR(k)) + \frac{A_1 - A_2}{1-s_w(1+n)}\frac{R(k)}{1+n}\pi_b$
Then, from combining i), ii), iii) it is clear that the three instrument-specific stability regions cannot be identical and that for all three instruments there exist stabilizing feedback coefficients \( \pi_k \) and \( \pi_b \) which are not contained in the joint intersection. \( \square \)

**Proof of proposition 3:**
The proof considers, for simplicity, only the \( g \)-regime and the \( \eta \)-regime and shows that for a golden rule steady state with a sufficiently high debt the intersection of the two stability regions is empty. Specifically, a steady-state constellation is derived where

\[
\text{i) } p(0)_{\eta} < 1, \text{ ii) } p(1)_{\eta} > 0, \text{ and iii) } p(-1)_{\eta} > 0
\]

are not jointly satisfied. For easy reference, consider Figure 8 which plots the three conditions at equality. As a particularly tractable example, which is consistent with (A 1)-(A 4), consider \( F(K, L) = z K^\alpha L^{1-\alpha} \) and \( U(c, d) = \ln d \), with \( z > 0, \delta = 1, n > 0 \). Note that the preferences imply that all disposable wage income is saved, i.e. \( s_w = 1 \). Moreover, let \( \theta = 0 \) and \( g = \eta > 0 \), and steady-state lump-sum taxes are calibrated such that \( \eta = \eta_s \cdot z^\alpha, \eta_s \in (0, 1) \). We derive in three steps a critical value \( \bar{\pi} \in (0, 1) \) such that, whenever \( \alpha \in (0, \bar{\pi}) \), there exists a golden rule steady state with \( b > 0 \) at which parts i) and ii) of (48) and part iii) are not jointly satisfied.

**Step 1):** Consider a steady state with \( b = 0 \) and \( k = k_0 > 0 \), satisfying

\[
k_0 = \frac{(1-\alpha)z k_0^\alpha - \eta}{1+n} = \frac{(1-\alpha-\eta_s)z k_0^\alpha}{1+n} \Leftrightarrow k_0 = \left[\frac{(1-\alpha-\eta_s)z}{1+n}\right]^{1/\alpha}.
\]

A golden rule steady state satisfies

\[
1 + n = R(k) = \alpha z k_{gr}^{\alpha-1} \Leftrightarrow k_{gr} = \left(\frac{\alpha z}{1+n}\right)^{\frac{1}{\alpha-1}}.
\]

Hence, \( b_{gr} > 0 \) at the golden rule steady state if \( k_0 > k_{gr} \Leftrightarrow \alpha < \frac{1-\eta_s}{2} = \bar{\alpha}_0 \). Note that \( b_{gr} + k_{gr} = \frac{1}{1+n}(1-\alpha-\eta_s)z k_0^\alpha \), implying \( b_{gr} = (\frac{1-2\alpha-\eta_s}{\alpha}) \cdot k_{gr}, R'(k_{gr}) \cdot b_{gr} = (1+n)(\frac{\alpha-1}{\alpha})(1-2\alpha-\eta_s), A_1 = 1 + n, \text{ and } A_2 = w'(k_{gr}) = (1+n)(1-\alpha) \), where we use that \( \eta \) is not a proportional, but rather a lump-sum tax.

**Step 2** Consider Figure 8. Then one can show that there exists a value \( \bar{\pi}_1 \in (0, \bar{\pi}_0) \) such that \( p(0)_{\eta} = 1 \) and \( p(1)_{\eta} = 0 \) slope as in Figure 8 and have an intersection with coordinates \( \pi_b^{**} > 0 \) and \( \pi_k^{**} < 0 \) if \( \alpha \in (0, \bar{\pi}_1) \). To see this, one obtains from

\[\text{At the expense of more tedious algebra, the proof can be extended to the more general utility function } U(c, d) = \ln c + \beta \ln d, \text{ with } s_w = \beta/(1 + \beta).\]
above, using \( s_w = 1 \),

\[
p(1) |_{\eta} > 0 \iff \pi_k < R'(k_{gr}) \cdot b_{gr} + \frac{R'(k_{gr})b_{gr} + A_1 - A_2}{1 + n} \pi_b
\]

with \( \pi_k^- = R'(k_{gr}) \cdot b_{gr} < 0 \)

\[
p(0) |_{\eta} < 1 \iff \pi_k > A_2 - A_1 + \frac{R'(k_{gr})b_{gr} - A_2}{1 + n} \pi_b
\]

with \( \pi_k^+ = A_2 - A_1 < 0 \).

Moreover, upon substituting out, the intersection of the two conditions at equality has coordinates

\[
\pi_{b}^{**} = A_2 - A_1 - R'(k_{gr}) \cdot b_{gr} = \pi_k^+ - \pi_k^-
\]

\[
\pi_{k}^{**} = R'(k_{gr}) \cdot b_{gr} - \frac{(\pi_k^+ - \pi_k^-)^2}{1 + n} < 0
\]

Hence, \( \pi_{b}^{**} > 0 \iff \pi_k^+ > \pi_k^- \), and for the particular specification one obtains

\[
\pi_{b}^{**} > 0 \iff -\alpha(1 + n) > (1 + n)\frac{(\alpha - 1)}{\alpha}(1 - 2\alpha - \eta_s)
\]

\[
\iff \alpha^2 - (3 - \eta_s)\alpha + 1 - \eta_s > 0. \quad (49)
\]

Note that this condition is always satisfied for \( \alpha \) being sufficiently close to zero. Moreover, the roots of this quadratic expression at equality are given by \( \overline{\alpha} = 0 + 1 \pm \sqrt{(\overline{\alpha})^2 + 1} \). Let \( \overline{\alpha} = 0 + 1 - \sqrt{(\overline{\alpha})^2 + 1} \), which satisfies \( \overline{\alpha} \in (0, \overline{\alpha}) \). Then, if \( \alpha \in (0, \overline{\alpha}) \), Figure 8 obtains in terms of the \( \eta \)-regime.

**Step 3** : Finally, it is shown that there exists a value \( \overline{\alpha} \in (0, \overline{\alpha}) \) such that the intersection of \( p(0) |_{\eta} = 1 \) and \( p(1) |_{\eta} = 0 \) is to the southeast of \( p(-1) |_{\eta} = 0 \) if \( \alpha \in (0, \overline{\alpha}) \). Equivalently, we show that, if \( \alpha \in (0, \overline{\alpha}) \), \( \pi_b^*(\pi_{b}^{**}) > \pi_k^*(\pi_{k}^{**}) \), ruling out that all three conditions of (48) can be jointly satisfied. To see this, note that

\[
p(-1) |_{\eta} > 0 \iff \pi_k > R'(k_{gr}) \cdot b_{gr} - 2(A_1 + A_2) + \frac{A_1 + A_2}{1 + n} \pi_b,
\]

sloping upward at equality. Evaluating \( p(-1) |_{\eta} = 0 \) at \( \pi_{b}^{**} \) yields

\[
\pi_k^* = R'(k_{gr}) \cdot b_{gr} - 2(A_1 + A_2) + \frac{A_1 + A_2}{1 + n} (A_2 - A_1 - R'(k_{gr})b_{gr}),
\]

and

\[
\pi_k^* > \pi_{k}^{**} \iff \left( \frac{A_2 - A_1 - R'(k_{gr})b_{gr}}{1 + n} \right)^2 + \frac{A_1 + A_2}{1 + n} \cdot \frac{A_2 - A_1 - R'(k_{gr})b_{gr}}{1 + n} = \frac{2(A_1 + A_2)}{1 + n} > 0.
\]
Substituting the expressions for the particular specification one obtains

\[ \pi_k^* > \pi_k^{**} \iff \left( \frac{\alpha^2 - (3 - \eta_s)\alpha + 1 - \eta_s}{\alpha} \right)^2 + (2 - \alpha) \cdot \frac{\alpha^2 - (3 - \eta_s)\alpha + 1 - \eta_s}{\alpha} > 4 - 2\alpha. \]

From the discussion of (49), the LHS of this expression declines continuously in \( \alpha \) in the interval \( \alpha \in (0, \alpha_1) \), with \( LHS(\alpha) \to \infty \) if \( \alpha \to 0 \) and \( LHS(\alpha_1) = 0 \). The RHS declines linearly in \( \alpha \), with \( RHS(0) = 4 \) and \( RHS(\alpha_1) > 0 \). Hence there exists a value \( \alpha \in (0, \alpha_1) \) such that \( \pi_k^* > \pi_k^{**} \) if \( \alpha \in (0, \alpha) \). \( \square \)

**Appendix 2: Endogenous labour supply and distortionary taxes**

**Key features of the linearized dynamics under all three instruments:**

As derived in the main text, consider the equations (40)-(42)

\[
(1 + n)(k_{t+1} + b_{t+1}) = s\tilde{w}(k_t, \eta_t) - \eta_t + \frac{\theta_{t+1}}{R(k_{t+1}, \eta_{t+1})} \cdot R(k_{t+1}, \eta_{t+1}) - \frac{\theta_{t+1}}{R(k_{t+1}, \eta_{t+1})}
\]

\[
(1 + n)b_{t+1} = R(k_t, \eta_t)b_t - \pi_t
\]

\[
\pi_t = \eta_t - \frac{\theta_t}{1 + \eta_t} - g_t
\]

and let \( A_1^* = 1 + n - R_k \cdot [s_R + (1 - s_w)\frac{\theta}{R(k)}] \) and \( A_2^* = s_w\tilde{w}_k \). For variations in \( g_t \) and \( \theta_{t+1} \) the taxation term \( \eta_t \) will be held constant. Hence, because of the structural similarity to the analysis given in Appendix 1, it is clear that the characteristic polynomials in these two cases are given by

\[
p(\lambda)|_g = \lambda^2 - \frac{A_2^*}{A_1^*} + \frac{R(k) - \pi_b}{1 + n} - \frac{R'(k)b - \pi_k}{A_1^*} \cdot \lambda + \frac{A_2^*}{A_1^*(1 + n)}(R(k) - \pi_b)
\]

and

\[
p(\lambda)|_\theta = \lambda^2 - \theta_1^* + \theta_2^*
\]

\[
\theta_1^* = \frac{A_2^*}{A_1^*} + \frac{R(k) - \pi_b}{1 + n} - \frac{R'(k)b - \pi_k}{A_1^*} + \frac{(1 - s_w)\frac{1 + n}{R(k)}\pi_b(R'(k)b - \pi_k)}{A_1^*(1 + n)}
\]

\[
\theta_2^* = \frac{A_2^*}{A_1^*(1 + n)}(R(k) - \pi_b),
\]

with:

\[ \tilde{A}_1^* = A_1^* - (1 - s_w)\frac{1 + n}{R(k)}\pi_k \]
Regarding \( \eta \), the linearized dynamics, similar to (17) and (18) and by using \( \eta_k = \pi_k \) and \( \eta_b = \pi_b \), can be summarized as

\[
(A_1^* - \psi_1 \pi_k) \cdot d\kappa_{t+1} + (1 + n - \psi_1 \pi_b) \cdot dB_{t+1} = (A_1^* + \psi_2 \pi_k) \cdot d\kappa_t + \psi_2 \pi_b \cdot dB_t
\]

\[
\psi_1 = R_\eta [s_R (1 - s_w) \frac{\theta}{R(k)^2}]
\]

\[
\psi_2 = s_w (\tilde{w}_\eta - 1)
\]

and

\[
(1 + n) \cdot dB_{t+1} = [(R_k + R_\eta \pi_k)b - \pi_k] \cdot d\kappa_t + [R(k) - \pi_b + R_\eta \pi_b] \cdot dB_t,
\]

giving rise to the characteristic polynomial

\[
p(\lambda)_{\eta} = \lambda^2 - \zeta_1^* \lambda + \zeta_2^* \tag{50}
\]

\[
\zeta_1^* = \frac{A_1^* + \psi_2 \pi_k}{A_1^* - \psi_1 \pi_k} + \frac{R(k) - \pi_b + R_\eta \pi_b}{1 + n} - \frac{[(R_k + R_\eta \pi_k)b - \pi_k] + n - \psi_1 \pi_b}{(A_1^* - \psi_1 \pi_k)(1 + n)}
\]

\[
\zeta_2^* = \frac{(A_2^* + \psi_2 \pi_k)[R(k) - \pi_b + R_\eta \pi_b] - [(R_k + R_\eta \pi_k)b - \pi_k] \psi_2 \pi_b}{(A_1^* - \psi_1 \pi_k)(1 + n)}
\]

Note that (50) reduces to (47) if \( R_\eta = 0 \) (and, hence, \( \psi_1 = 0 \)) and \( \tilde{w}_\eta = 0 \) and if one replaces \( A_1^* \) and \( A_2^* \) by \( A_1 \) and \( A_2 \).

**Derivation of (45) and (46):**

Consider \( \varphi(l) = \frac{\xi}{1+\chi} l^{1+\chi} \), with \( \varphi'(l) = \xi l^\chi \). At the steady state, \( w = \varphi'(l) \), since \( \tau_t = 0 \). Hence, \( \tilde{w} = l \varphi'(l) - \varphi(l) = \frac{\xi}{1+\chi} l^{1+\chi} \). Using \( l_\eta(k, \eta) = \frac{1}{l[\varphi'(l) - F_{LL}(k, l)]} < 0 \), (38) and (39) turn into:

\[
R_\eta = F_{KL} \cdot l_\eta = \frac{-F_{KL}(k, l)}{l \cdot [\varphi'(l) - F_{LL}(k, l)]} < 0
\]

\[
\tilde{w}_\eta = l \cdot F_{LL} \cdot l_\eta = \frac{-F_{LL}(k, l)}{\varphi'(l) - F_{LL}(k, l)} > 0.
\]

Consider \( F(K, L) = z K^\alpha L^{1-\alpha} \), with \( F_L(k, l) = z(1-\alpha) k^\alpha l^{-\alpha} \). Using the steady-state relationship \( \varphi'(l) = w = F_L(k, l) \) one obtains upon differentiation, \( R_\eta = -\frac{\alpha}{\alpha + \chi} \cdot \frac{1}{k} \) and \( \tilde{w}_\eta = \frac{\alpha}{\alpha + \chi} \in (0, 1) \). \( \square \)
Figure 1: Underaccumulation steady state

Stability triangle associated with \( g_t \)-regime (example 1)
**Figure 2:** Underaccumulation steady state

*Adjustment paths under $g_t$—regime for different feedback coefficients (example 1)*

- **B:** dashed (blue) line
- **C:** solid (black) line
- **D:** dashed-dotted (red) line

![Graph showing response of $k_t$ and $b_t$ over time for different feedback coefficients](image)
Figure 3: Underaccumulation steady state

Stability regions for all three instruments (example 1)

Shaded area: common intersection

- $g_t$-regime: solid (green) line
- $\eta_t$-regime: dashed (blue) line
- $\theta_{t+1}$-regime: dashed-dotted (red) line
**Figure 4:** Golden rule steady state

*Stability regions for all three instruments (example 2)*

Shaded area: common intersection

- $g_t$-regime: solid (green) line
- $\eta_t$-regime: dashed (blue) line
- $\theta_{t+1}$-regime: dashed-dotted (red) line
Figure 5: Golden rule steady state

Stability regions for all three instruments (example 3)

- $g_t$-regime: solid (green) line
- $\eta_t$-regime: dashed (blue) line
- $\theta_{t+1}$-regime: dashed-dotted (red) line

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The fact that one side of the $\eta$-triangle (corresponding to $p(-1)|_{\eta} = 0$) falls exactly onto a demarcation line of the $\theta$-type stability region (corresponding to $p(-1)|_{\theta} = 0$) is not generic but caused by the particular numerical assumption $\beta = 1$ (i.e. $s.w. = 1$), as one can verify from the conditions stated in Appendix 1. If $\beta < 1$, $p(-1)|_{\eta} = 0$ slopes upward, while $p(-1)|_{\theta} = 0$ slopes downward. Conversely, if $\beta > 1$, $p(-1)|_{\eta} = 0$ slopes downward and $p(-1)|_{\theta} = 0$ slopes upward.
**Figure 6**: Golden rule steady state under distortionary taxation

*Stability regions for all three instruments (example 4)*

- $g_t$-regime: solid (green) line
- $\eta_t$-regime: dashed (blue) line
- $\theta_{t+1}$-regime: dashed-dotted (red) line
Figure 7: Golden rule steady state under distortionary taxation

Stability regions for all three instruments (example 5)

Shaded area: common intersection

$g_t$-regime: solid (green) line
$\eta_t$-regime: dashed (blue) line
$\theta_{t+1}$-regime: dashed-dotted (red) line
**Figure 8:** Golden rule steady state

*Empty intersection of $g_t$-triangle and $\eta_t$-triangle (Proof of Proposition 3)*

$g_t$-regime: solid (green) line  
$\eta_t$-regime: dashed (blue) line

Shaded areas represent necessary conditions for stability under the two regimes.