ENDOGENOUS HETEROGENEITY IN STRATEGIC MODELS:

SYMMETRY-BREAKING VIA STRATEGIC SUBSTITUTES

AND NONCONCAVITIES

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November 2008

Abstract

This paper is an attempt to develop a unified approach to endogenous heterogeneity and symmetry-breaking by constructing two general classes of two-player symmetric games that always possess only asymmetric pure-strategy Nash equilibria. These classes of games are characterized in some abstract sense by two general properties: payoff non-concavities and some form of strategic substitutability. Our framework relies on easily verified assumptions on the primitives of the game, which are satisfied in a number of two-stage models with an investment decision preceding product market competition. To illustrate the generality and wide scope for application of our approach, we present some existing models dealing with R&D and capacity expansion, which motivated this study.

Keywords: submodular games, asymmetric Nash equilibrium, inter-firm heterogeneity, supermodular games.

JEL Classification: C72, C62, L11.

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1 Introduction

When using symmetric games in applied work, analysts often restrict attention to symmetric equilibrium points even when other asymmetric outcomes exist and reflect equally or more pertinent behavior. In most cases, the only justification beyond simplicity is what Schelling (1960) convincingly termed the focal nature of symmetric equilibrium points. Yet, it is widely recognized that inter-agent heterogeneity is often a critical dimension of various economic and social phenomena. From a positive perspective, heterogeneity is a necessary postulate to account for the simple fact that in the real world, one seldom observes identical agents, be it individuals, firms, industries or countries. From a normative standpoint, differences across interacting agents often constitute a necessary condition for many key economic activities such as trade and risk-sharing.

Understanding the origins and evolution of diversity across economic agents or disparities in economic performance across economies is increasingly perceived as a central goal of economic and social research in different areas. Macroeconomists seek to explain the causes of booms and recessions. Development economists grapple with the forces behind poor and strong economic performances. Labor economists attempt to get a handle on discriminatory treatment of groups of workers. Business strategists and industrial economists devote a lot of attention to the sources and sustainability of inter-firm heterogeneity within and across industries. Overall, much effort has been expanded with a view to explain “diversity across space, time and groups” (Matsuyama, 2002).

In view of the diversity of economic research areas involved in this effort, several conceptual and methodological approaches have been developed. While often tailored to a specific area, each of these approaches is broad in explanatory scope and has wide potential applicability.

The dominant approach, based on coordination failures, postulates a game with strategic complementarities and multiple Pareto-ranked pure-strategy Nash equilibrium points. Diversity is then synonymous with making different equilibrium selections, with the high-performing entity picking the Pareto-dominant
equilibrium and the low-performing entity failing to do so. This argument is thus generally predicated on the presence of two identical and non-interacting economies, each operating under a different equilibrium out of the same equilibrium set. It may also be invoked to explain diversity across time within the same economy, with booms and recessions corresponding to operation under the Pareto dominant and inferior equilibria respectively. See Cooper and John (1988), Murphy, Schleifer and Vishny (1989), and Cooper (1999).

The coordination failures approach has been criticized for failing to offer any compelling argument for the diversity in equilibrium selections in the two-economy model or for the regime switch in the one-economy model. Matsuyama (2002) proposes an interactive link between the two sub-economies by allowing the two a priori identical players to take two decisions, one in each sub-economy. Assuming players’ actions are pairwise strategic complements and each player’s own actions are substitutes due to a fixed total resource constraint, multiple equilibria arise with the property that the symmetric equilibria are Cournot-unstable while the asymmetric ones are stable. Endogenous heterogeneity in this original approach is then predicated on the key postulate that only Cournot-stable equilibria are observable outcomes of this complex game, any of which would involve each agent taking different actions in the two ex ante identical sub-economies. In his pioneering work, Matsuyama (2000, 2002, 2004) coined the term “symmetry-breaking” to refer to this heterogeneity-generating process.

The third approach originates in the business strategy literature, which has a long-standing tradition of research on firms as complex organizations and on the sources and dynamics of inter-firm heterogeneity. In the dominant view, firms operate in such highly complex and ever-changing environments that they entertain no hope of ever accumulating enough knowledge about their world to view it as a strategic game or formulate a precise strategy. Rather, firms grope for economic performance via a heuristic learning process involving idiosyncratic trial and error and the continual updating of routines and rules of thumb eschewing optimization. Heterogeneity is then simply an inevitable outcome of such processes, and firms end up with different heuristic strategies and core
capabilities (Nelson and Winter, 1982).

The present paper is an attempt to contribute to this debate along standard lines of argument in applied game theory and industrial organization. Consider a two-player symmetric normal-form game characterized by two key properties: actions form strategic substitutes, and each player’s payoff, though continuous, admits a robust concavity-destroying kink along the diagonal in action space, which results in a jump of the reaction correspondence across the 45° line. Such a game always admits pure-strategy Nash equilibrium points due simply to the property of global strategic substitutes. Furthermore, due to the jump over the diagonal, no such equilibrium could ever be symmetric. At any of the possibly multiple equilibria, which occur in pairs due to the symmetry of the game, otherwise identical agents will necessarily take different equilibrium actions. With this being the main result of the paper, the same conclusion is also shown to hold for a class of games with payoffs that are quasi-convex in own action and submodular on the four-point square formed by the players’ extreme actions only, reflecting thus global nonconcavity and localized submodularity.

The present paper may also be motivated in relation to various broad strands of literature in industrial economics dealing in some way with strategic endogenous heterogeneity along lines similar to ours here. The first literature that comes to mind is concerned with product differentiation. In a myriad of two-stage games where each firm chooses a quality level or a horizontal characteristic in the first stage, and then a price for its product in the second stage, endogenous heterogeneity naturally emerges out of the firms’ perception that identical choices in the first stage will lead to zero profits in the second stage Bertrand competition due to the resulting homogeneity of the products.¹

The second, extensive literature deals with infinite-horizon industry dynamics allowing for entry and exit. One class of models, exemplified by Jovanovic (1982), postulates perfectly competitive firms for which differences emerge due to exogenous idiosyncratic technology shocks. Another class is formed by studies

that do generate endogenous heterogeneity in long run dynamics by considering firms that invest in capacity expansion (e.g. Besanko and Doraszelski, 2002) or R&D (e.g. Doraszelski and Satterthwaite, 2004). Some simpler two-stage models with similar flavor but without entry and exit also generate endogenous differences amongst competing firms: Maggi (1996), Reynolds and Wilson (2000) and De Frutos and Fabra (2007) for capacity expansion and Mills and Smith (1996) and Amir and Wooders (2000) for R&D.

There are other studies in various areas of applied microeconomics where endogenous heterogeneity emerges in a strategic setting. A partial list follows. Hermalin (1994) develops a two-stage game where firms’ choices of managerial structures take place before market competition. Mills and Smith (1996) and Amir (2000) deal with R&D/product market competition games giving rise to equilibrium outcomes with maximal heterogeneity only, i.e. full R&D by one firm and no R&D by the rival.2 In a tax competition model, Mintz and Tulkens (1986) exhibit asymmetric tax rates for identical member states.

Each of these studies might be construed as a context-specific attempt to shed light on to the central issue at hand. Taken all together, one might hope that these studies share some general features or a common mechanism for generating symmetry breaking in strategic settings, which could be put forth as another systematic approach. As an alternative motivation, the present paper is an attempt to develop a unifying approach to understanding symmetry-breaking mechanisms in general classes of two-player games, encompassing many of the cited studies. These two-stage models share two key features that are critical for the symmetry-breaking arguments they present. The first is a fundamental nonconcavity in the payoffs, which may be confined to the diagonal in action space or hold globally, and the second is some form of strategic substitutes in first-period actions. A noteworthy aspect of the results is that they rely on

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2 Another strand of literature, not directly related to our setting, deals with endogenous heterogeneity arising out of hybrid models of joint ventures where firms make a cooperative decision in the first stage followed by strategic product market competition in the second stage: See Salant and Shaffer (1999) and Long and Soubeyran (2001).
critical assumptions that arise naturally, are economically intuitive and easy to verify in various applications presented below.

The paper is organized as follows. Section 2 contains the overall set-up. Section 3 provides the results on the exclusive existence of asymmetric equilibria for submodular payoff functions and Section 4 presents the results for games with quasi-convex payoffs. Each section provides a summary of the relevant applications the results pertain to. The Appendix provides most of the proofs.

2 The set-up

This section lays out the set-up and general notation for use throughout the paper. Consider a two-player normal form symmetric game \( \Gamma \) with common action set \( X = [0, c] \subset \mathbb{R} \), and payoff functions \( F \) and \( G : X \times Y \to \mathbb{R} \). By symmetry of the game \( \Gamma \), the payoff of player 2 is \( G(x, y) = F(y, x) \). Except where otherwise indicated, the payoff function of player 1 is always of the form

\[
F(x, y) = \begin{cases} 
U(x, y), & x \geq y \\
L(x, y), & x < y 
\end{cases}
\]

for some functions \( U \) and \( L \) with respective domains \( \Delta_U = \{(x, y) \in [0, c]^2 : x \geq y\} \) and \( \Delta_L = \{(x, y) \in [0, c]^2 : x \leq y\} \).

It will be assumed throughout this section that \( U \), \( L \) and \( F \) are jointly continuous functions of the two actions, so in particular that \( U(x, x) = L(x, x) \), \( \forall x \in [0, c] \). It follows that the best response correspondences (or reaction curves) for players 1 and 2, defined respectively as \( r_1(y) = \arg \max \{F(x, y) : x \in [0, c]\} \) and \( r_2(x) = \arg \max \{F(y, x) : y \in [0, c]\} \) are well-defined.

As usual, a pure strategy Nash equilibrium, (or PSNE for short), \( (x^*, y^*) \in [0, c]^2 \) is said to be symmetric if \( x^* = y^* \), and asymmetric otherwise. It follows from the symmetry of the game that if \( (x^*, y^*) \) is a PSNE, so is \( (y^*, x^*) \).

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3The noncooperative game described may be a simple one-shot game or it may represent the payoffs of a two-stage game as a function of the first period actions, where the unique
The next two sections investigate two separate classes of normal-form symmetric games that always possess asymmetric Nash equilibria and no symmetric Nash equilibria. For each of the two classes, we provide a set of easily verified assumptions establishing both the existence of asymmetric, and the inexistence of symmetric, PSNEs. To illustrate the relevance of our results and the ease of verification of our sufficient conditions, we provide some detailed illustrations based on previous studies in industrial organization where a special case of our results was derived in a specific economic setting.

The definitions and main results from the theory of supermodular games used in this paper are reviewed in the appendix in an elementary way, which is sufficient for the purposes of this paper.

3 Endogenous heterogeneity with strategic substitutes and diagonal non-concavity

In this section, we consider a two-player normal-form symmetric game characterized by two key properties. The first is that, conditional on one player using a higher action than the other, the two actions form strategic substitutes. This means that an increase in one player’s strategy lowers the other player’s marginal returns to increasing his own strategy, as long as all the latter’s actions remain larger or smaller than the former’s. The second key property is that each player’s payoff, though jointly continuous in the two actions, admits a fundamental nonconcavity along the 45° line, giving rise to a canyon shape along the diagonal. A key consequence of this feature is that a player would never optimally respond to an action of the rival by playing that same action himself.

Taken together, these two properties imply that the actions are globally strategic substitutes, and thus that each best reply is a decreasing corresponding second stage pure-strategy equilibrium has been substituted in. In the latter case, which actually covers most of the applications of this paper, we obviously restrict consideration to subgame-perfect equilibria and analyze the resulting one-shot game.
dence, which in addition has a downward jump over the 45° line. It follows that a pure-strategy Nash equilibrium (henceforth, PSNE) exists, see Appendix. Hence, no PSNE could ever be symmetric. At any of the possibly multiple equilibria, which occur in pairs due to the symmetry of the game, ex ante identical agents will necessarily take different equilibrium actions.

3.1 The results

Different subsets of the following assumptions will be needed for our conclusions below. The notation is as laid out in Section 2. A full discussion of the assumptions and results is presented at the end of the section. Most of the proofs can be found in the Appendix.

A1 $U$, $L$ are twice continuously differentiable on $\Delta_U$ and $\Delta_L$, respectively, and $F$ is continuous along the diagonal, i.e. $U(x,x) = L(x,x), \forall x \in [0,c]$.

A2 $U$, $L$ are strictly submodular on the sublattices $\Delta_U$ and $\Delta_L$, respectively.

A3 $U_1(x^+, x) > L_1(x^-, x), \forall x \in (0,c)$

A4 $U_1(0^+, 0) > 0$ and $L_1(c^-, c) < 0$

A2 says that on either side of the diagonal, but not necessarily globally, each player’s marginal returns to increasing his action decrease with the rival’s action. A3 holds that each player’s payoff, though globally continuous in the two actions, has a kink along the diagonal in the shape of a canyon. The role of A4 is simply to rule out PSNEs at $(0,0)$ or $(c,c)$.

We say that a function $f : R \rightarrow R$ is increasing (strictly increasing) if $x' > x$ implies $f(x') \geq (>) f(x)$. A correspondence is increasing if all its selections are increasing functions.

In addition, throughout the paper, partial derivatives are denoted by a subindex corresponding to the relevant variable, i.e. $U_1(x, y) = \frac{\partial U(x,y)}{\partial x}$ and $U_2(x, y) = \frac{\partial U(x,y)}{\partial y}$. One-sided derivatives at a point are indicated by a + or − sign as an exponent after the point, e.g. $U_1(0^+, y)$ is the partial derivative from the right w.r.t $x$ at $(0,y)$.

On the boundaries of the domains $\Delta_U$ and $\Delta_L$, the derivatives are to be understood as one-sided (directional) derivatives.

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6On the boundaries of the domains $\Delta_U$ and $\Delta_L$, the derivatives are to be understood as one-sided (directional) derivatives.
These assumptions form a sufficiently general framework to encompass many of the studies mentioned in Section 1 as special cases. Furthermore, all the assumptions are easy to check directly on the primitives of a particular game.

The following is the main result of this paper.

**Theorem 3.1** Assume that $A_1 – A_4$ hold. Then the game $\Gamma$ is of strategic substitutes, has at least one pair of asymmetric PSNEs and no symmetric PSNEs.

The idea of the proof is that global submodularity of the payoff function is inherited from the submodularity of its components $U$ and $L$ in the presence of assumption $A_3$, which leads by Topkis’s monotonicity theorem to globally decreasing best replies. $A_3$ ensures that the best response correspondences must have a downward jump that crosses over the diagonal.\(^7\)

The prototypical case is depicted in Figure 1.\(^8\) Global submodularity of $F$ gives us the existence of a PSNE via the strategic substitutes property, and the jump across the diagonal (at point $d$) precludes symmetric equilibria. That PSNEs come in pairs is a direct consequence of the symmetry of the game. The complete proof can be found in the Appendix.\(^9\)

Towards the goal of generating endogenous heterogeneity, the present approach is closest in spirit to Matsuyama’s symmetry-breaking explanation. By allowing for suitable discontinuities in the players’ reaction curves, it dispenses

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\(^7\)This result would be more general if one could use the dual single-crossing property instead of submodularity (Milgrom and Shannon, 1994). There are two reasons that justify our choice of submodularity. The first is that all the economic applications that motivated the present paper use objective functions that are formed by adding separate parts (such as revenue and costs), an operation that need not preserve the single-crossing property. The second reason is that our approach for showing that the overall payoff inherits the submodularity of the separate components $U$ and $L$ does not appear to extend to the single-crossing property.

\(^8\)Note that $x$ is the variable on the vertical axis throughout. This corresponds to analyzing the game from the point of view of player 1 choosing $x$ as a response to $y$. Also, continuity of the reaction curves in each triangle over and below the diagonal is only there for the sake of a clearer figure. It need not hold under Assumptions $A_1$- $A_4$.

\(^9\)The proof that the overall payoff inherits the submodularity of $U$ and $L$ in the presence of assumption $A_3$ is surprisingly long. The key difficulty is that Topkis’s cross-partial test cannot be invoked, due to the kink along the diagonal.
Figure 1: Decreasing reaction curves have a jump along the diagonal, and there is no symmetric equilibrium

with the need to interconnect two separate economies in the subtle and somewhat complex manner proposed by Matsuyama. More importantly, it provides a framework that is independent of the somewhat controversial argument of outright rejecting Cournot-unstable equilibria. Indeed, even when one ignores the focal nature derived from their symmetry, one should keep in mind that these equilibria cannot be ruled out on account of any of the Nash equilibrium refinements, such as normal-form perfection or strategic stability (Kohlberg and Mertens, 1986).\textsuperscript{10}

There is some experimental evidence can shed light on this issue. In a laboratory setting involving a symmetric Cournot duopoly with one unstable symmetric equilibrium and a pair of asymmetric equilibria, Cox and Walker (1998) found little support in the data for \textit{any of the three equilibria}. This provocative finding suggests that while a Cournot-unstable equilibrium of a given game may be justifiably regarded as unobservable in economic reality, it does not thereby follow that some Cournot-stable equilibrium of the same game will necessarily prevail in actual play of the game and thus be observable.

\textsuperscript{10}Indeed, they are typically strict Nash equilibria (in the sense that a unilateral deviation will lead to a strict loss for the deviator, as opposed to indifference), and thus would survive any of the well-known Nash equilibrium refinements.
Rather, the presence of both Cournot-stable and Cournot-unstable PSNEs may well engender a high level of intrinsic indeterminacy, which may critically reduce the predictive power of the Nash concept altogether for such games, as in the Cox-Walker experiment.

Theorem 3.1 does not rule out the existence of multiple pairs of PSNEs either. Indeed, the two reaction curves may intersect several times above and below the diagonal. In case of multiple pairs of PSNEs, there will typically be co-existence of pairs of Cournot-stable and pairs of Cournot-unstable PSNEs. Nevertheless, Theorem 3.1 does imply that all of these PSNEs are asymmetric. Thus, the present approach to symmetry-breaking is not at odds with Schelling’s (1960) notion of focalness of PSNEs.

Since payoffs are jointly continuous in actions for this class of games, a symmetric mixed-strategy Nash equilibrium always exists (Dasgupta and Maskin, 1986). As this would be the only focal equilibrium in the sense of Schelling (1960), it may reasonably be advanced as a plausible outcome of such a game. Nevertheless, in actual realizations of the equilibrium randomizations, the players will still end up actually playing different actions with strictly positive, though not full, probability. Hence, given our focus on explaining observed heterogeneity, the approach followed in the present paper need not rule out mixed strategies a priori. On the other hand, since two-player submodular games are also supermodular games, all mixed-strategy equilibria are unstable for a large class of natural learning dynamics (Echenique and Edlin, 2004).

The next result adds further restrictions, of a dominant diagonal type, on the payoff components of our game that lead to a unique pair of PSNEs, which are then necessarily Cournot-stable. In this case, symmetry-breaking is coupled with more predictive power, although the selection of one PSNE from the pair still remains indeterminate, as is standard in symmetric settings.

11 For comparative statics properties of such equilibria, see Echenique (2004).

12 There is also a symmetric mixed-strategy equilibrium here (although \( U \) and \( L \) are strongly concave, the overall payoff function is not concave). Hence, this result is not in conflict with the well-known results on the odd number of total (pure and mixed) Nash equilibria.
Theorem 3.2 Assume that $A1 - A4$ are in effect, and the following holds:

\[
\begin{align*}
U_{11}(r(z), z) - U_{12}(r(z), z) &< 0 \quad (2) \\
L_{11}(r(z), z) - L_{12}(r(z), z) &< 0, \quad (3)
\end{align*}
\]

then the game admits exactly one pair of PSNEs.

The proof of this result establishes that, under (2) and (3), the restrictions of each reaction curve to the sets \( \Delta_U \) and \( \Delta_L \) are both (decreasing) contractions, so that within each of these sets, there exists a unique PSNE. Because the diagonal-skipping jump at the point \( d \) is still present, the reactions curves are still discontinuous at \( d \), and are thus clearly not contractions over all of \([0, c]\).

Another perspective is that while (2) and (3) ensure that \( U \) and \( L \) are each quasi-concave in own action, the overall payoff \( F \) does not inherit the same property, due to the fundamental non-concavity along the diagonal. Likewise, the property of diagonal dominance of \( U \) and \( L \), i.e. (2) and (3), which is key to our limited uniqueness result, is not inherited by the overall payoff function \( F \). By contrast, submodularity of \( U \) and \( L \) is inherited by \( F \), which is the key property behind the existence of PSNEs.

The next result is devoted to comparing the two asymmetric PSNEs from any given pair from the point of view of the players’ welfare. Given any pair of asymmetric PSNEs, it is often of interest to determine circumstances under which a given equilibrium secures better payoffs for a player. In other words, under what conditions would each player prefer the PSNE where he is the high or the low-activity player? To this end, we need to impose a condition of monotonicity on the payoff function of the player in question along his opponent’s best reply as stated in the following result, which lays out conditions for player 1 (say) to prefer the PSNE where he is the low-activity player.

Theorem 3.3 Let \((x^*, y^*)\) and \((y^*, x^*)\) be equilibria in \( \Delta_U \) and \( \Delta_L \), respectively, so that \( x^* > y^* \). If \( A1 - A4 \) hold and moreover \( U(r_1(y), y) \) and \( L(r_1(y), y) \) are increasing in \( y \in [0, d] \) and \( y \in [d, c] \) respectively, then \( F(x^*, y^*) \leq F(y^*, x^*) \).
This monotonicity assumption, that $U$ and $L$ are increasing in rival’s action along the player’s best response, is well-defined in the sense that $U(r_1(y), y)$ and $L(r_1(y), y)$ are single-valued even though $r_1$ can be multi-valued, in $y$. This assumption is clearly more general than assuming that, for each fixed action of a player, his payoff is increasing in the rival’s action. For a simple example illustrating this point, see von Stengel (2003).

The obvious dual statement giving conditions under which each player would prefer the high-activity equilibrium is omitted.

### 3.2 Some applications

In this section we present examples of economic models that constitute special cases of the general framework developed above. While the assumptions validating Theorem 3.1 might at first appear somewhat special, they are satisfied in several a priori unrelated studies that have established endogenous heterogeneity in strategic settings. Going over some of these examples illustrates the unifying character and the ease of application of our results, and allows us to provide some contextual interpretations of symmetry-breaking.\(^{13}\)

#### 3.2.1 R&D investment

In Amir and Wooders (2000), two a priori identical firms with initial unit cost $c$ are engaged in a two stage game of R&D investment and production. In the first stage, autonomous cost reductions $x$ and $y$ for firms 1 and 2, respectively, are chosen. The novel feature of this study is that spillovers are postulated to flow only from the more R&D active firm to the rival, but not vice versa. The effective (post-spillover) cost reductions $X$ and $Y$ when $x \geq y$ are given by:

$$X = x \text{ and } Y = \begin{cases} x \text{ with probability } \beta \\ y \text{ with probability } 1 - \beta \end{cases}$$

\(^{13}\)In some studies, asymmetric equilibria are produced via a mechanism similar to our Theorem 3.1, without being a special case in a formal sense, e.g. Hermalin (1994).
Second stage product market competition, be it Cournot or Bertrand, is assumed to have a unique PSNE with equilibrium payoffs given by \( \Pi : [0, c]^2 \to R \). \( \Pi(x, y) \) is the payoff of the firm whose unit cost is the first argument. Let \( f : [0, c] \to R \) be a known R&D cost schedule, with \( f'(x) \geq 0 \). Assume the following (in addition to smoothness of \( \Pi \) and \( f \)):

**C1** \( \Pi \) is strictly submodular and \( \Pi_1 (x, y) < 0 \) and \( \Pi_2 (x, y) > 0 \)

**C2** \(|\Pi_1(x, x)| > |\Pi_2(x, x)|\), \( \forall x \in [0, c] \)

**C3** \( f'(0) < -\beta\Pi_2(c, c) - \Pi_1(c^+, c) \) and \( f'(c) = -(1 - \beta) \Pi_1(0^+, 0) \).

The overall payoff of firm 1, \( F(x, y) \), defined as in (1), is given by the difference between its second stage profit and first stage R&D cost.

\[
U(x, y) = \beta \Pi(c - x, c - y) + (1 - \beta) \Pi(c - x, c - y) - f(x) \quad (5)
\]

\[
L(x, y) = \beta \Pi(c - y, c - y) + (1 - \beta) \Pi(c - x, c - y) - f(x) \quad (6)
\]

We can easily check that \( A_1 \) and \( A_2 \) indeed hold. \( U(x, x) = L(x, x) \), \( \forall x \in [0, c] \), so \( F \) is continuous. \( A_1 \) can be checked by using the cross-partial test and the fact that \( \Pi(x, y) \) is submodular (C1). Using C1\(-\)C2, and

\[
U_1(x, x) = -[\Pi_1(c - x, c - x) + \beta\Pi_2(c - x, c - x)] - f'(x) \quad (7)
\]

\[
L_1(x, x) = -(1 - \beta) \Pi_1(c - x, c - x) - f'(x) \quad (8)
\]

we see that \( U_1(x, x) > L_1(x, x) \), so \( A_3 \) holds. Finally, \( A_4 \) follows from C3.

Hence, Theorem 3.1 applies directly to this model. In addition, the uniqueness of a pair of asymmetric PSNEs is shown by imposing plausible conditions that secure that the reaction curves are contractions, so that Theorem 3.2 can be applied. Similarly, extra conditions are needed to apply Theorem 3.3 and conclude that firm 1 prefers the equilibrium in which it is the high R&D firm (for details, see Amir and Wooders, 2000).

The key driving force behind asymmetric equilibrium outcomes here is the one-way nature of the spillover process. A firm will always react by performing either less R&D than its rival knowing that it may free ride on the difference
in R&D levels, or, in case the rival’s R&D is simply too low, by overtaking it. In this vision, firms will endogenously settle into R&D innovator and imitator roles simply as a reflection of the nature of the R&D spillover process.

3.2.2 Capacity choice and demand uncertainty

De Frutos and Fabra (2007) offer an elegant and general analysis of a common two-period model where firms choose capacities under demand uncertainty in the first period, and then engage in price competition upon demand realization in the second period. When demand has a continuous distribution, they establish the existence of an asymmetric PSNE, with no symmetric PSNE being possible, under general assumptions. When demand has a discrete distribution, symmetric and asymmetric PSNE may co-exist, but the latter are always there. Thus, under quite general conditions, their model leads to endogenous capacity asymmetries even though firms are ex-ante identical. De Frutos and Fabra provide some interesting intuition behind the twin incentives that lead identical firms to settle into high and low capacity producers. The underlying mechanism under continuous demand relies on the same two key properties and is thus quite closely parallel to our main result. Their analysis might alternatively be done by checking $A_1 - A_4$. Finally, Reynolds and Wilson (2000) have a related result, but in a more specific setting.

3.2.3 Provision of information

Ireland (1993) presents a two-stage model of information provision in a Bertrand oligopoly. In the first stage, firms decide on the provision of information about the existence of their product, and in the second stage, they compete in prices. Each firm has monopoly power over the consumers who are only informed about the existence of its product and not of the rival’s product. As such, whenever the information coverage choice of one firm is low, the other wishes to opt for a high information coverage. Hence, profit functions are submodular. The non-concavity in the diagonal arises from price competition driving profits to zero.
whenever the same consumers are informed of the existence of both products.
This paper concludes that no symmetric PSNE in information provision exists, and that two asymmetric PSNE may be found. It is trivial to show that assumptions $A1 - A4$ are verified in this example and could be used to obtain the asymmetry result.

4 Games with quasi-convex payoffs

In this section we analyze a class of symmetric games in which payoff functions are strictly quasi-convex in own action for each fixed rival’s action. This leads players to always prefer one of their extreme actions in response to any strategy of the rival, i.e. no action in $(0,c)$ could ever be a best response to any pure action by the rival. It follows that only asymmetric PSNEs involving extreme actions can arise. A dual result is in Amir et al. (2008), who consider $n$-player symmetric games on chains with order-quasi-convex payoffs that are supermodular for extreme actions, and show existence of symmetric equilibria only.

In this section, the payoff function $F$ need not be of the $U/L$ form used in the rest of the paper, so that notation will not be used. None of the assumptions on the game from previous sections, including smoothness, are in effect here, unless so stated explicitly.

Given $y$, define the right lower Dini derivate w.r.to $x$ at 0 and the left upper Dini derivate w.r.to $x$ at $c$ respectively by (recall that these derivates always exist in the extended reals for any function, Royden, 1968)

$$F_1(0^+, y) \triangleq \liminf_{x \downarrow 0} \frac{F(x, y) - F(0, y)}{x}$$
and
$$F_1(c^-, y) \triangleq \limsup_{x \uparrow c} \frac{F_1(c, y) - F(x, y)}{c - x}$$

Theorem 4.1 Consider a symmetric game with action set $[0,c]$ and payoff function $F(x, y)$ that is upper semi-continuous and strictly quasi-convex in own action. Assume that either

$$F(c, 0) > F(0, 0) \text{ and } F(0, c) > F(c, c)$$
(9)
or

\[ F_1(0^+, 0) > 0 \text{ and } F_1(c^-, c) < 0 \]  \hspace{1cm} (10)

Then, the game has no symmetric PSNE and it has exactly one pair of asymmetric PSNEs given by \{ (0, c), (c, 0) \}.

**Proof.** Since \( F(x, y) \) is upper semi-continuous in \( x \), the reaction correspondence \( r_1(y) \) is well-defined. By definition of strict quasi-convexity,

\[ F(\lambda z_1 + (1 - \lambda) z_2) < \max\{ F(z_1), F(z_2) \}, \forall z_1, z_2 \in [0, c]^2 \text{ with } z_1 \neq z_2 \]

Taking \( z_1 = (0, y) \) and \( z_2 = (c, y) \), for some \( y \in [0, c] \), we know that against any action \( y \in [0, c] \) by Player 2, any \( x \in (0, c) \) yields a strictly lower payoff to Player 1 than either \( x = 0 \) or \( x = c \). This means that \( \forall y \in [0, c], r_1(y) \in \{0, c\} \), i.e. \( r_1(y) = 0 \) or \( r_1(y) = c \) or both. By symmetry, the same conclusion holds for player 2.

Now consider (9). Clearly, \( r_1(0) \in \{0, c\} \) and \( F(c, 0) > F(0, 0) \Rightarrow r_1(0) = c \). Likewise, \( r_1(c) \in \{0, c\} \) and \( F(0, c) > F(c, c) \Rightarrow r_1(c) = 0 \). Hence \( (c, 0) \) and \( (0, c) \) are the only PSNEs of the game.

Finally, consider (10). From the definition of strict quasi-convexity, it follows that \( F_1(0^+, 0) > 0 \) implies that \( F(\cdot, 0) \) is strictly increasing on \([0, c]\), and thus that \( r_1(0) = c \). Likewise, strict quasi-convexity and \( F_1(c^-, c) < 0 \) implies that \( F(\cdot, 0) \) is strictly decreasing, and thus \( r_1(c) = 0 \). Hence \( (c, 0) \) and \( (0, c) \) are the only PSNEs of the game.

In actual applications, it is often easier to check condition (10) than condition (9), since the former is often a direct consequence of natural Inada-type assumptions on the primitives of a model, usually assumed to be smooth. On the other hand, for models with non-specific functional forms, checking (9) directly is often difficult.

Submodularity in extreme actions is clearly implied by (9). Indeed, adding up the two inequalities in (9) reveals that \( F \) is submodular on the 4-point system \{ (0, 0), (0, c), (c, 0), (c, c) \}, although \( F \) need not have any complementarity property on \((0, c)^2\).
Figure 2: Quasi-convexity of payoffs implies players prefer corner solutions.

As the only continuity assumption on the payoff is upper semi-continuity w.r.t. own action only, the reaction correspondences need not have the closed graph property (or upper hemi-continuity here). This property is actually not relevant to the existence argument invoked here (in contrast to the classical approach based on Kakutani’s fixed point theorem).

In addition to the pair of PSNEs, the game will also admit one symmetric mixed strategy equilibrium with support \{0, c\}, so that maximal endogenous heterogeneity, i.e. the equilibria outcomes (0, c) and (c, 0), prevail with positive but not full probability. This equilibrium in unstable for a broad class of learning dynamics (Echenique and Edlin, 2004).

With weak instead of strict quasi-convexity, the PSNE set would still include (0, c) and (c, 0), but possibly other points as well, so endogenous heterogeneity need not prevail, unless some equilibrium selection arguments are introduced.

Figure 2 illustrates Theorem 4.1.

This result clearly extends to asymmetric games, with the same proof being valid in that setting. Since the focus of this paper is on endogenous heterogeneity, we dealt with the symmetric version. Indeed, the players may have different payoff functions and action sets \([0, c_1]\) and \([0, c_2]\), in which case the result is that the profiles (0, c_2) and (c_1, 0) are the only PSNEs.
We now list a number of existing applications of this result. The best-known models that use the same arguments as the present result are those dealing with duopoly with vertically differentiated products (Gabszewicz and Thisse, 1979 and Shaked and Sutton, 1982). Consider two firms engaged in a two-stage game where they make a quality decision in the first stage and a price decision in the second stage. Under a few variants of such a basic set-up, the overall payoff of each firm as a function of the two quality decisions, conditional on a Bertrand equilibrium in the price subgame, is strictly convex in own quality. Since the same quality choice would lead to homogeneous goods in the price subgame and thus zero profit, subgame-perfect equilibrium calls for the adoption of maximal quality by one firm and minimal quality by the other firm (the quality space is an exogenous compact interval). It is trivial to verify that the analysis presented in these related papers is a special case of the present result.\footnote{While the literature on vertical differentiation has restricted attention to symmetric duopoly, the fact that equilibrium quality levels lie at the two extremes extends to asymmetric models. This is so for the same reason that Theorem 4.1 extends to asymmetric games.}

Theorem also generalizes the results of Mills and Smith (1996) and Amir (2000), who consider two-stage duopoly games of R&D/product market competition. In the first stage, firms make R&D investment decisions that affect the production costs. In the second stage, firms compete à la Cournot. Under some plausible assumptions, each profit function is strictly convex in own R&D decision, so that extreme asymmetric outcomes form the only equilibria of the game. It is easily shown that the conditions presented in these papers can also be deduced from the assumptions of Theorem 4.1.

5 Conclusion

This paper has provided simple, easily verified and general conditions on the primitives of a symmetric game that ensures that heterogeneity in a priori identical agents’ behavior necessarily arises. This paper constitutes, hence, a contribution to the discussion on the sources of diversity across economic agents and
disparities in economic performances. While previous literature stands on arguments related to distinct equilibrium selections and strategic complementarities (Cooper, 1999) or on a mix of strategic substitutability and complementarity and Cournot-stability of equilibria (Matsuyama, 2002), our approach stands on the existence of a fundamental nonconcavity along the diagonal of the payoff function and on some form of strategic substitutability. The ensuing symmetry breaking is simpler and arguably more unambiguous than in previous approaches. The scope for economic applications is quite broad, as demonstrated by the many existing studies that the present analysis was conceived to unify.

6 Appendix

6.1 Summary of supermodular/submodular games

We give an overview of the main definitions and results in the theory of supermodular games that are used in this paper, in a simplified setting that is sufficient for our purposes. Details may be found in Topkis (1978, 1998).\(^{15}\)

Let \(I_1\) and \(I_2\) be compact real intervals. \(F : I_1 \times I_2 \to \mathbb{R}\) is (strictly) supermodular\(^{16}\) if \(\forall x_1, x_2 \in I_1, x_2 > x_1\) and \(\forall y_1, y_2 \in I_2, y_2 > y_1\) we have \(F(x_2, y_2) - F(x_2, y_1) > F(x_1, y_2) - F(x_1, y_1)\). \(F\) is (strictly) submodular if \(-F\) is (strictly) supermodular.

Let \(F\) be twice continuously differentiable. Then \(F\) is supermodular (submodular) if and only if \(F_{12} = \frac{\partial^2 F}{\partial x \partial y} \geq 0 \ (\leq 0)\) for all \(x, y\).

If \(F_{12} = \frac{\partial^2 F}{\partial x \partial y} > 0 \ (< 0)\) for all \(x, y\), then \(F\) is strictly supermodular (submodular).

We now present a dual (and special case) of Topkis’s monotonicity theorem, which is suitable for our use in this paper.

\(^{15}\)Other aspects of the theory that are relevant to the present paper may be found in Topkis (1979), Vives (1990), Milgrom and Roberts (1990) and Amir (1996a), among others.

\(^{16}\)More precisely, this is actually the standard definition of (strict) increasing differences in the literature. For functions on \(\mathbb{R}^2\), increasing (decreasing) differences is equivalent to supermodularity (submodularity). We shall use only the latter terminology throughout.
Theorem 6.1 If $F$ is continuous in $x$ and (strictly) submodular in $(x, y)$, then $\arg\max_{x \in I_1} F(x, y)$ has maximal and minimal (all of its) selections that are decreasing in $y \in I_2$.

A two-player game is supermodular (submodular) if both payoff functions are continuous, supermodular (submodular) and both action spaces are compact real intervals. The fixed point theorem associated with this framework is due to Tarski (1955).

Theorem 6.2 Let $f : I_1 \times I_2 \rightarrow I_1 \times I_2$ be an increasing function, then $f$ has a fixed point.

This theorem enables one to prove that a supermodular game always has a pure strategy Nash equilibrium. A two-player submodular game becomes a supermodular game upon reversing the order on one player’s action set.

6.2 Proofs of Section 3

The proof of Theorem 3.1 is organized as follows: we begin with proving four preliminary lemmas, and then present the main proof in two steps: first we show existence of PSNE and afterwards that all PSNEs must be asymmetric. Recall that, contrary to standard practice, the variable $x$ is on the vertical axis while $y$ is on the horizontal axis.

The first lemma states that for a small enough square of points in $[0, c]^2$ with two vertices on the diagonal, strict submodularity of $F$ holds.

Lemma 6.1 Let $(x, x) \in (0, c)^2$ and consider the four points depicted in Figure 3. If $A_1 - A_3$ hold, then for $\alpha > 0$ small enough,

$$L(x, x) - L(x - \alpha, x) < U(x, x - \alpha) - U(x - \alpha, x - \alpha)$$

(11)

or equivalently,

$$F(x, x) - F(x - \alpha, x) < F(x, x - \alpha) - F(x - \alpha, x - \alpha).$$
Figure 3: Square of length $\alpha$.

**Proof.** Take any diagonal point $(x, x) \in (0, c)^2$. It follows from $A3$ and $A1$ that given $\varepsilon = U_1(x^+, x) - L_1(x^-, x)$, there is $\alpha > 0$ such that for all $\alpha < \alpha$, 

$$L(x - \alpha, x - \alpha) - L(x - 2\alpha, x - \alpha) + \varepsilon/2 \leq U(x, x - \alpha) - U(x - \alpha, x - \alpha). \quad (12)$$

From $A2$ we know that

$$L(x - \alpha, x) - L(x - 2\alpha, x) < L(x - \alpha, x - \alpha) - L(x - 2\alpha, x - \alpha). \quad (13)$$

Since $L_1(x, y)$ is continuous in $x$ for fixed $y$ (from $A1$), it follows that given the above $\varepsilon$, there is $\overline{\alpha} > 0$ such that for all $\alpha < \overline{\alpha}$,

$$|(L(x - \alpha, x) - L(x - 2\alpha, x)) - (L(x, x) - L(x - \alpha, x))| < \varepsilon/2 \quad (14)$$

Recapitulating, for all $\alpha < \overline{\alpha} \land \alpha$, we have

$$L(x, x) - L(x - \alpha, x) < L(x - \alpha, x) - L(x - 2\alpha, x) + \varepsilon/2 \text{ by } (14)$$

$$< L(x - \alpha, x - \alpha) - L(x - 2\alpha, x - \alpha) + \varepsilon/2 \text{ by } (13)$$

$$\leq U(x, x - \alpha) - U(x - \alpha, x - \alpha) \text{ by } (12)$$

The inequality with the two outer terms yields (11). ■

The next lemma extends the property of submodularity of $F$ from small squares to any square with two vertices on the diagonal.
Lemma 6.2 For any square with two vertices, \((z, z)\) and \((x, x)\), on the diagonal, such as depicted in Figure 4, we have, with \(x > z\),

\[ F(x, x) - F(z, x) < F(x, z) - F(z, z). \]

**Proof.** Consider the square depicted in figure 4 and divide it into rectangles, each of which has height equal to the original height of the square and length not larger than \(\alpha\), as defined in Lemma 6.1. We will now show that \(F\) is supermodular for the four-point system formed by the vertices of each such rectangle, and then we extend the conclusion to the whole square.

Let \(x - z = k\alpha\), where \(k \in \mathbb{R}\), and \(\alpha > 0\) is small enough. Now consider the rectangle defined by the vertices \((x, x)\), \((x, x - \alpha)\), \((z, x)\) and \((z, x - \alpha)\). From Lemma 6.1 we know that

\[ F(x, x - \alpha) - F(x - \alpha, x - \alpha) > F(x, x) - F(x - \alpha, x) \]

Also, from A2 we know that

\[ F(x - \alpha, x - \alpha) - F(z, x - \alpha) > F(x - \alpha, x) - F(z, x) \]

since all the points belong to \(\Delta_L\). Adding these two inequalities we obtain that

\[ F(x, x - \alpha) - F(z, x - \alpha) > F(x, x) - F(z, x) \]

(15)

Repeating the procedure, consider the rectangle defined by: \((x, x - \alpha)\), \((x, x - 2\alpha)\), \((z, x - \alpha)\), \((z, x - 2\alpha)\). From A2 we know that:

\[ F(x, x - 2\alpha) - F(x - \alpha, x - 2\alpha) > F(x, x - \alpha) - F(x - \alpha, x - \alpha) \]

since all the points belong to \(\Delta_U\). Likewise,

\[ F(x - 2\alpha, x - 2\alpha) - F(z, x - 2\alpha) > F(x - 2\alpha, x - \alpha) - F(z, x - \alpha) \]

since all the points belong to \(\Delta_L\). Using Lemma 6.1 we know that

\[ F(x - \alpha, x - 2\alpha) - F(x - 2\alpha, x - 2\alpha) > F(x - \alpha, x - \alpha) - F(x - 2\alpha, x - \alpha) \]

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Adding the three inequalities we obtain:

\[
F(x, x-2\alpha) - F(z, x-2\alpha) > F(x, x-\alpha) - F(z, x-\alpha) > F(x, x) - F(z, x) \text{ by (15).}
\]

We can repeat this argument \(k\) times until we get a rectangle whose width is not bigger than \(\alpha\). Once again we use \(A2\) and Lemma 6.1 to show that submodularity holds for this rectangle as well and

\[
F(x, z) - F(z, z) \geq F(x, x) - F(z, x).
\]

Hence submodularity holds for any square with 2 vertices on the diagonal. ■

The following Lemma establishes that the analysis of submodularity on any four-point rectangle in \([0, c]^2\) can be reduced to the analysis of submodularity on squares with 2 vertices on the diagonal.

**Lemma 6.3** If \(A1\) and \(A2\) hold, then \(F\) is submodular on \([0, c]^2\).

**Proof.** Due to the kink along the diagonal, one cannot invoke Topkis’s simple cross-partial test to verify submodularity of \(F\). Instead, we use the definition of submodularity for any four-point rectangle in \([0, c]^2\). If the rectangle is completely contained in either \(\Delta_U\) or \(\Delta_L\), then submodularity follows from \(A2\).
Figure 5: If $F$ satisfies submodularity on the square on the diagonal, this implies it satisfies submodularity on the rectangle.

Every other situation can be reduced by adding sub-rectangles, each of which lying fully in either $\Delta_U$ or $\Delta_L$, to the situation depicted in Figure 3 as we now show. Consider the case of Figure 5 with the four points $(x, z), (z, z), (x, y), (z, y)$ as shown. With $z < x < y$, we know from A2 that, since $F(x, y) = U(x, y)$ on $\Delta_U$, we have

$$F(x, x) - F(z, x) > F(x, y) - F(z, y).$$

From Lemma 6.2, submodularity holds for the vertices of the square $(x, x), (x, z), (z, z), (z, x)$, hence we have

$$F(x, z) - F(z, z) > F(x, x) - F(z, x).$$

Adding the two inequalities yields

$$F(x, z) - F(z, z) > F(x, y) - F(z, y),$$

which is just the definition of submodularity for $(x, z), (z, z), (x, y)$ and $(z, y)$.

It can be shown via analogous steps that the submodularity of $F$ for any other configuration of four points can be reduced to showing submodularity on squares with two vertices on the diagonal. The details are left out. ■

The next result allows us to conclude that the two reaction curves always
admit a unique discontinuity that skips over the diagonal, a key step for our endogenous heterogeneity result.

**Lemma 6.4** Given $A_1 - A_3$, there exists exactly one point $d \in (0, c)$, such that $r_i(d^-) > d > r_i(d^+), i = 1, 2$.

**Proof.** From Topkis’s monotonicity theorem and Lemma 6.3, all the selections from the best reply correspondences are decreasing. From the general properties of monotone functions, both the right limit $r_i(x^+)$ and the left limit $r_i(x^-)$ exist at any point $x$ for any selection of $r_i$, and are independent of the selection. Specifically, for any selection $r_i$, $r_i(x^+) = \overline{r}(x)$ and $r_i(x^-) = \underline{r}(x)$, where $\overline{r}$ and $\underline{r}$ denote the maximal and minimal selections of $r_1$.

From assumption $A_3$, we know that $(0,0) \not\in \text{Graph } r_i$ and $(c,c) \not\in \text{Graph } r_i$ (i.e. $r_i$ does not go through $(0,0)$ or $(c,c)$). These two properties imply that $r_i$ cannot be identically $0$ or $c$.

We next show that the reaction correspondence $r_1$ cannot ever intersect the $45^\circ$ line at an interior point, i.e. $(0,c)$. The generalized first order condition for a maximum of $F$ (say) to occur at a point $(x,x)$ with $x \in (0,c)$, which applies even in the absence of differentiability, is that $U_1(x^+,x) \leq L_1(x^-,x)$.

Assumption $A_3$ rules out this possibility. Hence no $x \in (0,c)$ can ever be a best reply to itself, meaning that the reaction correspondences do not cross the $45^\circ$ line at any interior point.

Since $r_1$ starts strictly above $0$ (for $y = 0$) and ends strictly below $c$ (for $y = c$), the above properties of $r_1$ imply that there exists exactly one $d \in (0,c)$ such that $r_1(d^-) = \overline{r}(d) > d > r_1(d^+) = \underline{r}(d)$. In words, $r_1$ must have a downward jump that skips over the diagonal as in Figure 1. \hfill \blacksquare

**Proof of Theorem 3.1.**

From Lemma 6.3 we have the global submodularity of the payoff function. This guarantees that a PSNE exists.

Consider now the behavior of the reaction curves in the area $\Delta_U$. The same conclusion follows for $\Delta_L$ by symmetry. It is easy to verify that (any
Equivalently, since to Amir (1996b) and Edlin and Shannon (1998), or see [Topkis, 1998, p. 79].

shown in the previous proof. To show that all the slopes of the restricted mappings \( r_1|_{\Delta_v} (y) \) and \( r_2|_{\Delta_v} (x) \) take values in the shown ranges when restricted to the given domains. Also, both restricted mappings are decreasing as implied by Lemma 6.3. Define the mapping \( B: [d, c] \rightarrow [d, c], B(x) = r_1|_{\Delta_v} \circ r_2|_{\Delta_v} (x) \), which is increasing given that each of \( r_1|_{\Delta_v} \) and \( r_2|_{\Delta_v} \) is decreasing. From Tarski’s fixed point theorem, we know that there exists \( \bar{x} \) such that \( B(\bar{x}) = r_1|_{\Delta_v} \circ r_2|_{\Delta_v} (\bar{x}) \), therefore \( (\bar{x}, r_2 (\bar{x})|_{\Delta_v}) \) is a PSNE. Hence, there must exist at least one pair of asymmetric PSNEs. From Lemma 6.4, there is no symmetric PSNE in \([0, c]^2\).

Proof of Theorem 3.2.

We show that there is a unique PSNE in the rectangle \([0, d] \times [d, c]\) by showing that the restricted mappings \( r_1|_{\Delta_v} (y) \) and \( r_2|_{\Delta_v} (x) \) as defined in the previous proof intersect exactly once in the rectangle \([0, d] \times [d, c]\), as their slopes are all confined to \((-1, 0]\). The same conclusion follows for rectangle \([d, c] \times [0, d]\) by symmetry, so that the overall game admits exactly one pair of PSNEs.

Clearly, the slopes of \( r_1|_{\Delta_v} (y) \) are \( \leq 0 \) since \( r_1 \) is globally decreasing, as shown in the previous proof. To show that all the slopes of \( r_1|_{\Delta_v} (y) \) are \( > -1 \), consider a change of decision variable for Player 1 given by \( z = x + y \). This leads to the equivalent best-response problem on \([0, d] \times [d, c]\) being \( \max_z \{ U(z - y, y) : z \in [2y, y + c], y \in [0, d]\} \), instead of \( \max_z \{ U(x, y) : x \in [y, c], y \in [0, d]\} \). The partial \( \partial U(z - y, y)/\partial z = U_1(z - y, y) \) is strictly increasing in \( y \) if \( \partial^2 U(z - y, y)/\partial z \partial y = -U_{11} + U_{12} > 0 \), which is our assumption (2). Furthermore, the constraint set \([2y, y + c]\) is ascending in \( y \). Hence, every selection of the argmax \( z^*(y) \) is strictly increasing in \( y \), by a strengthening of Topkis’s Theorem due to Amir (1996b) and Edlin and Shannon (1998), or see [Topkis, 1998, p. 79]. Equivalently, since \( z^*(y) = r_1|_{\Delta_v} (y) + y \), every selection of \( r_1|_{\Delta_v} (y) \) has all its slopes \( > -1 \). (Note that \( r_1|_{\Delta_v} (y) \) may be equal to \( c \) for an initial interval of values of \( y \), but cannot be equal to \( y \), as shown in the previous proof. This possible boundary behavior is clearly consistent with our arguments here.) Overall then, the slopes of \( r_1|_{\Delta_v} (y) \) are all in \((-1, 0]\), which also implies that \( r_1|_{\Delta_v} (y) \)
is single-valued.

The fact that there is a unique PSNE in $\Delta_U$ follows from a known argument, similar to the Contraction Mapping Theorem applied to the mapping $B$ (for details, see e.g. Amir, 1996a).\textsuperscript{17} By symmetry, the overall game admits exactly one pair of PSNEs.

Finally we provide a proof of Theorem 3.3.

**Proof of Theorem 3.3.**

Since $x^* > d > y^*$, we have $F(x^*, y^*) = U(x^*, y^*) = U(r_1(y^*), y^*)$ and $F(y^*, x^*) = L(y^*, x^*) = L(r_1(x^*), x^*)$. Also $U(r_1(d), d) = L(r_1(d), d)$ if $d$ denotes the unique jump point across the diagonal, as defined in Lemma 6.4. Then

$$F(x^*, y^*) = U(r_1(y^*), y^*)$$
$$\leq U(r_1(d^-), d) \text{ since } U(r_1(y), y) \text{ is increasing in } y$$
$$= L(r_1(d^+), d) \text{ by definition of } d$$
$$\leq L(r_1(x^*), x^*) \text{ since } L(r_1(y), y) \text{ is increasing in } y$$
$$= F(y^*, x^*).$$

**References**


\textsuperscript{17}Note that the Contraction Mapping Theorem does not quite apply here, as the contraction property requires that that all the slopes of the mapping at hand be $\leq \beta$, for some $\beta < 1$. 


