MORTGAGE VALUATION
A Quasi-Closed Form Solution

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A MODEL FOR MORTGAGE VALUATION: A QUASI-CLOSED FORM SOLUTION

SUMMARY

The main objective of this study is to develop a quasi-analytical solution to allow for the valuation of commercial mortgages. For that, we consider the existence of only one source of risk, the risk of defaulting on a mortgage, and therefore, the existence of only one state variable, the value of the mortgaged property. The value of the mortgage corresponds to the present value of the future payments on the loan, minus the value of the default option. The major difficulty in designing this model lies in calculating the value of this option, since for that purpose it is necessary to determine the critical value of the property, i.e. the lowest price below which the option must be immediately exercised.

With the introduction of these limitations, the partial differential equation for the value of the default option has a similar configuration to that presented by Black and Scholes (1973). However, the boundary conditions applicable to the value of the default option are different, due to the possibility of early default. To obtain a quasi-closed form solution, instead of continuous time framework, a discrete time setting is considered for the differential equation, (approach usually known as the ‘Method of Lines’). The partial differential equation is transformed into an ordinary non-homogenous differential equation, replacing the derivative of the function [price of the option in relation to time] with a finite difference, and keeping the derivatives of the mortgaged property unchanged. In order to improve the performance of the model, Richardson’s extrapolation is applied, allowing for the increase in the speed of convergence of the model results towards reasonable values.

Working out a quasi-closed solution for this type of financial asset allows the presentation of an alternative valuation framework whereby obtaining results, as well as performing sensitivity analysis of the mortgage price in relation to other elements in the model, can be much quicker when compared to the alternative use of numerical techniques. This work represents one of the first attempts to address the development of analytical solutions to mortgage valuation, using contingent claims analysis.

Key words: Mortgage valuation; option valuation; default; quasi-closed solution.
1. Introduction

Contingent claims mortgage valuation models tend to assume that the value of these financial assets is mainly determined by one or two sources of risk: risk of prepayment and risk of default. In turn, these sources of risk are normally modelled through the consideration of two stochastic variables: the interest rate and the value of the mortgaged property.

The possibilities held by the borrower to end the respective loan, either by way of prepayment or by way of default represent sources of uncertainty that are difficult to model. The decision of the borrower is conditioned simultaneously by the future evolution of both, interest rates and prices of the mortgaged property. The prepayment option on a mortgaged asset is equivalent to an American call option on a bond whose cash flows are equivalent to the outstanding payments associated with the mortgage loan. In turn, the default option can be considered as an American put option, in which the underlying asset is the mortgaged property².

The contingent claims valuation models that take into consideration both mortgage options allow for a better understanding of the characteristics and components of mortgage contracts. This is important not only academically, but also for its practical implications, since the pricing of assets tends to naturally improve as a result of different investment banks progressively adopting these types of models.

In the overwhelming majority of cases, the specialist literature has presented only numeric solutions for this type of evaluation. In order to obtain these solutions, complex calculation techniques are required, which means the process becomes overly slow and expensive.

² It could also be seen, probably even more aptly, as a European compound option i.e. as a series of European call options with payments made on each of the due dates inherent in the loan (see, for example, Kau, Keenan, Muller and Epperson, 1995, and Azevedo-Pereira, Newton and Paxson, 2002, 2003).
In this area of research, an aspect that has still not been explored much is the development of closed form solutions for valuation of mortgage assets. The mathematical complexity that is provoked by the simultaneous existence of the two American options has, up to now, impeded the development of efficient valuation models.

With the aim of making a contribution towards surpassing these difficulties, the main objective of this study is the development of a quasi-closed form solution that will allow the valuation of commercial mortgages. In order to achieve this goal, to make the modelling possible, it is necessary to impose a set of limitations on our study. Therefore, the mortgage valuation model proposed in this work does not take into consideration the prepayment option. Consequently, interest rates are not regarded as stochastic. Although the inclusion of these limitations tends to move the model away from reality, the work is intended as a first step in developing analytic solutions for mortgage valuation – a process that it is hoped to undergo successive improvements during the coming years.

Under these assumptions, the value of the mortgage will correspond to the present value of the future payments minus the value of the default option. The present value of future payments is obtained by applying the concept of present value of an annuity, since the interest rate is not stochastic. The value of the default option is determined using the principles that normally underlie the valuation of American put options.

Following this approach, the valuation of a commercial mortgage starts with the definition of the stochastic process used to characterize the behaviour of the stochastic variable considered in the model, the value of the mortgaged property. The most frequently used methodology is that defined by Black and Scholes (1973) and Merton (1973) for modelling the price of a European call option on a stock.

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3 One exception is Collin-Dufresne and Harding (1999) which proposes a closed solution for fixed rate residential mortgages, using a single variable, the short-term interest rate.
The next step consists of obtaining the partial differential equation for the value of the default option. This equation is, once more, similar to that presented by Black and Scholes (1973). However, the boundary conditions applicable to the value of the default option are different. The difference lies in the possibility early exercise of the default option by the borrower. In this sense, the main difficulty in designing the model lies in calculating the critical value of the property, i.e. the price level below which the option should be immediately exercised. To ensure the feasibility of reaching a quasi-closed solution, it is still necessary to transform the partial differential equation into an ordinary non-homogenous one. In order to achieve this result, following the approach proposed by Carr and Faguet (1996) for the valuation of American options, the Method of Lines (MOL) is applied. This means that the time derivative of the function [price of the option in relation to time] is replaced by a finite difference. The approximation error tends to reduce as the size of the time increment used in the MOL is reduced. In this way, to obtain a good approximation to the value of the option, it would be convenient to consider time as divided into a number of intervals tending towards infinity, which in practice is impossible to apply. In order to overcome this difficulty, Richardson’s extrapolation is applied. Richardson’s extrapolation is a technique that allows an accelerated convergence of results towards the reference values, despite the fact that time is considered as being broken down into a relatively lower number of sub-periods.

Under the previous assumptions, and considering the income of the mortgaged property as proportional to its value, it will be developed a commercial mortgage valuation model for which it is possible do derive a quasi-closed solution.

The outline for this paper is as follows. Section 1 introduced the work. Section 2 describes the main assumptions underlying the mortgage valuation model and presents the stochastic processes followed by corresponding variables, the general partial differential equation and the corresponding options held by the borrower. Section 3, develops the commercial mortgage valuation model, and presents its quasi-closed form solution. Some of the model’s limitations will also be referred to. In section 4, the formulae derived during the course of the work are verified and validated via graphical
and numerical analysis. Finally, the conclusion is presented, including suggestions for future developments.

2. The Mortgage Valuation Model

2.1. General Formulation of the Contingent Claims Mortgage Valuation Model

Some contingent claims mortgage valuation models include both risk factors inherent in a mortgage contract whilst others limit themselves to either the default or the prepayment option. To characterise these risk factors, two stochastic variables are usually used: the short-term risk-free interest rate, \( r \), and the value of the mortgaged property, \( B \).

The most used specification of short-term interest rate is probably that which was originally proposed by Cox, Ingersoll and Ross (1985):

\[
dr = k(\theta - r)dt + \sigma_r \sqrt{r} \, dz_r
\]  

(1)

Where:

- \( k \) ≡ Rate of adjustment
- \( \theta \) ≡ Long-term mean of \( r \)
- \( \sigma_r \) ≡ Instantaneous standard deviation
- \( z_r \) ≡ Standard wiener process

In studies of mortgage valuation, this framework is regarded as a reference for modelling risk related to the term structure of interest rates. Its performance in this type of structure was analysed by Archer and Ling (1995), and obtained good results compared with the no-arbitrage model of Black, Derman and Toy (1990). Over the past two decades, various mortgage valuation models have adopted this framework to characterise interest rate behaviour. Among them, it is important to refer to the work of Azevedo-Pereira, Newton

In turn, the stochastic process followed by the value of the property is identical to the process used by Black and Scholes (1973) and Merton (1973) to model the price of European call options on stocks:

$$dB = (\alpha - b)Bdt + \sigma_B Bdz_B$$  \hspace{1cm} (2)

Where:

- $\alpha \equiv$ Instantaneous revenue expected on the property
- $b \equiv$ Rate of continuous revenue on the property
- $\sigma_B \equiv$ Instantaneous standard deviation
- $z_B \equiv$ Standard Wiener process

On the other hand it is assumed that $dz_r dz_B = \rho_{rb} dt$, where $\rho_{rb}$ designates the coefficient of instantaneous correlation.


After the presentation of the stochastic processes followed by the variables, the next step consists of deriving the partial differential equation for the value of commercial mortgage, $V$, considering there are no arbitrage opportunities. The formulation of this equation is the result of the dynamics followed by both variables underlying the model, the short-term interest rate and the value of the property.

$$\frac{\partial V(\tau, B, r)}{\partial \tau} + \frac{\partial V(\tau, B, r)}{\partial r} k(\theta - r) + \frac{\partial V(\tau, B, r)}{\partial B} (r - b) B + \frac{1}{2} r \sigma_r^2 \frac{\partial^2 V(\tau, B, r)}{\partial r^2} +$$

$$+ \frac{1}{2} \sigma_B^2 B^2 \frac{\partial^2 V(\tau, B, r)}{\partial B^2} + \sigma_r \sqrt{\theta} \rho_{\theta} \sigma_B B \frac{\partial^2 V(\tau, B, r)}{\partial \theta \partial B} - r V(\tau, B, r) = 0$$

(3)

This general equation allows the valuation of financial derivatives whose underlying variables follow the previously defined stochastic processes. The difference in valuation formulae, for different financial assets, resides in the initial, final, and boundary conditions to be applied according to each concrete case. For mortgage valuation purposes, the works that have used (3) assuming the possibility of both default and of prepayment, have, in the majority of cases, presented numerical solutions. As far as the authors are aware, do not exist analytic solutions for valuing mortgages that satisfy (3) considering simultaneously the possibilities of default and of prepayment.

The stochastic processes used to characterize the behaviour of the variables in the general mortgage valuation model generate three possible trajectories in space $(r, B, t)$. One of the routes is the normal termination of a mortgage contract with all instalments paid in due time, in which case neither the option of prepayment nor of default is exercised.
Another possibility is represented by the prepayment boundary that exists during the life of the loan and that corresponds to situations of high property prices and low interest rates. Lastly, the default boundary is reached when property prices are sufficiently low. This can happen on each date, \( t = i \), on which a loan instalment is due. The prepayment and default boundaries should be exogenously determined in accordance to the rational behaviour of the borrower. We are going to assume that, at the moment of signing the loan contract, the mortgage cost for the borrower corresponds to the minimum that might be achieved (taking into consideration the value of the options to prepay and default\(^4\)).

2.2. The Model to Be Developed in this Work

2.2.1. General Characteristics

Given the inherent difficulties in considering the both early termination options simultaneously, this study will restrict its analysis to the valuation of a commercial mortgage with a default option. This assumption is in concordance with other work already undertaken in this area, such as Titman and Torous (1989). According to these authors, some types of commercial mortgages do not allow for the prepayment of the loan prior to its full term. In effect, the institutional characteristics tend to make that postulation more reasonable in relation to commercial mortgages than in relation to residential mortgages.

Faced with this assumption, the model becomes somewhat simpler. The value of the mortgage corresponds to the value of a financial asset whose only source of risk derives from the possibility of borrower default.

Kau and Keenan (1999) state that it is possible to find analytic solutions for mortgage valuation models that consider the risk of default if a reduced form of (3) - in which time and the short-term interest rates are not taken into consideration – is used. To justify the

\(^4\) For a more detailed explanation see, for example, Kau and Keenan (1999).
removal of the time period from the equation, the authors propose studying the value of a perpetual mortgage with a fixed rate.

Following on from that study, the model here developed assumes that interest rates are constant and that prepayment is not allowed. However, contrary to what was established by Kau and Keenan (1999), the time period is not eliminated from the differential equation, which creates the possibility of evaluating a mortgage with a finite maturity.

In this context, the value of the mortgage is given by the following formula:

\[ V(\tau, B) = A(\tau) - P(\tau, B) \]  

(4)

In this equation, \( A(\tau) \) equals the present value, in continuous time, of the future mortgage instalments:

\[ A(\tau) = C \left[ \frac{1 - e^{-r\tau}}{r} \right] \]  

(5)

Where, \( C \), represents the annual value of the (fixed) instalment, \( r \), the interest rate, and \( \tau \), the number of years remaining until the maturity of the mortgage.

The other component of (4), \( P(\tau, B) \), equals the value of an American put option whose underlying asset is the value of the mortgaged property, \( B \). The great difficulty in solving (4) lies in this part of the equation. The literature presents numerous studies on the valuation of American puts, but only for a few particular cases is possible to find analytic solutions. The problem becomes still more complicated when the underlying asset pays dividends. Unfortunately, that is the case modelled here, since properties provide revenue. In effect, the model assumes the existence of a revenue rate, \( b \), proportional to the value of the property, and similar to that included in (2).
2.2.2. The Default Option

At any moment in time, a borrower that defaults on mortgage instalments will lose his property. Consequently, a mortgage contract is equivalent to a American put option whose underlying asset is the value of the property and whose strike price is given by the present value of future instalments.

The value of this option is given by Black and Scholes’ (1973) partial differential equation:

\[
-\frac{\partial P(\tau, B)}{\partial \tau} + \frac{\partial P(\tau, B)}{\partial B}(r - b)B + \frac{1}{2} \sigma^2 B^2 \frac{\partial^2 P(\tau, B)}{\partial B^2} - rP(\tau, B) = 0
\]

For the valuation of a mortgage default option, this equation is subject to the following boundary conditions:

\[
\lim_{B \to \infty} P(B, \tau) = 0 \quad (7)
\]

\[
\lim_{B \to 2} P(B, \tau) = A(\tau) - B(\tau) \quad (8)
\]

\[
\lim_{B \to 2} \frac{\partial P(B, \tau)}{\partial B} = -1 \quad (9)
\]

As was previously mentioned, obtaining a closed solution for this problem is an extremely complex task. In spite of some advances made in the last few years it has not been possible to arrive at a consensual analytical solution.

The possibility of early default constitutes the main obstacle in determining a closed solution. Therefore, most of the works that present solutions for pricing these options use
numerical methods to determine the critical price of the underlying asset. For example, studies such as Carr, Jarrow and Myneni (1992), Jacka (1991) and Kim (1990) consider that the value of the American option is equal to the value of a European option, increased by a premium given by the present value of the gains induced by early termination. This premium is obtained through an integral equation, whose exercise boundary is determined by a reversive numerical procedure.

In turn, studies such as those by Lee and Paxson (2003), Huang, Subrahmanyam and Yu (1996), Ho, Stapleton and Subrahmanyam (1994), Bunch and Johnson (1992), and Geske and Johnson (1984), develop an analytical approximation for the price of an American put option by means of calculating the value of a set of options with discrete dates of exercise. The main difference between these studies lies in the number and exercise dates of the options used for calculating the value of the American put, as well as the extrapolation method used with the intention of obtaining a satisfactory approximation for the value of the option. However, these works also do not determine analytically the critical value of the underlying asset.

One of the improvements introduced in this field during the last decade consists of using the ‘Method of Lines’. The core idea behind this approach is to approximate the partial derivative in relation to time, in the corresponding differential partial equation, with a finite difference. In parallel, the partial derivatives in relation to the price of the underlying asset remain unaltered. The application of the procedure results in a sequence of ordinary non-homogenous differential equations. The approximation error is determined by the size of the time increments used in the finite difference representation. In order to accelerate the convergence of the error approximation towards zero, Richardson’s extrapolation may be applied.

5 The value of the European option is obtained according to the formula given by Black and Scholes (1973).
6 There are some works that use linear extrapolation, whilst others utilise exponential extrapolation.
7 In this case, the substitution is made in (6).
8 For an understanding of how the extrapolation is applied, see Carr (1998).
The MOL was used in option valuation frameworks by Carr and Faguet (1996) and by Meyer and J. van der Hoek (1997). The first of these works utilised the method to generate explicit formulae for the approximate value of an American option and the corresponding critical price. The second dealt with how the method might be used in the study and numerical valuation of American options and also as a means of determining the early exercise boundary.

Apart from Carr and Faguet (1996), another reference in the field of analytic solutions for American put options is Carr (1998). This article conceives time as random considering that it follows an Erlang or gamma distribution. Although the methods used to reach analytic solutions were different in Carr (1998) and Carr and Faguet (1996), the results obtained in the both articles were the same.

The present work follows Carr and Faguet (1996) and uses the MOL to pursue the outlined goal of obtaining a quasi-closed solution for mortgage valuation.

2.2.3. Commercial Mortgage Valuation Model: a Quasi-Closed Form Solution

In mortgage valuation models, application of the MOL has implications regarding the way in which the present value of future loan instalments is calculated. Additionally, it has also repercussions concerning the differential equation for the value of the built-in default option.

As far as the present value of future instalments is concerned, instead of using a continuous time formulation, it is necessary to use the corresponding formulae in discrete time.

Based on this alteration:
In this equation, \( C \), the annual value of the continuous instalments, is substituted by \( n \) constant discrete instalments whose value is equal to \( \frac{C \tau}{n} \). When the MOL is applied to calculate the value of the default option, the number of constant instalments should coincide with the number of intervals in which the time to maturity of the contract is subdivided.

In determining the value of the default option, application of the MOL comes down to substitution of the time derivative [price of the option in relation to time] with a finite difference. The smaller the time interval considered in the finite difference, the greater the accuracy of the results reached with this alteration. However, a high number of time intervals brings a significant increase in mathematical computation. Therefore, there is a tendency to use a relatively small number of time intervals\(^9\), usually four. Afterwards, Richardson’s extrapolation is normally applied to accelerate the convergence of results towards values closer to reality.

With the application of this method, the differential equation given in (6), takes on the following form, for \( B \geq B_k \):

\[
A = \frac{C}{r} \left( 1 - \frac{1}{\left( 1 + r \frac{\tau}{n} \right)^n} \right)
\]  

\(^9\) Considering the period to maturity of the contract as being divided into four time intervals gives a most acceptable estimate for the value of an American put option. In mortgage valuation, the same rationale is applied, since the valuation of a default option is conceptually similar to the valuation of an American put option.
\[
\frac{\partial P^{(k)}(B)}{\partial B}(r - b)B + \frac{1}{2}\sigma_{\sigma}^2 B^2 \frac{\partial^2 P^{(k)}(B)}{\partial B^2} - rP^{(k)}(B) = \frac{P^{(k)}(B) - P^{(k-1)}(B)}{\tau/n}
\tag{11}
\]

To calculate \(P^{(k)}(B)\) it is necessary to determine in advance the expression for \(P^{(k-1)}(B)\), which implies that this equation should be solved for \(k = 1,2,3,4\) \(^{10}\). This equation is somewhat simplified in relation to equation (6), since the price of the option becomes a function of a single variable, the value of the mortgaged property. Given this alteration, the general expression for the value of the mortgage contract is to the following:

\[
\nu^{(k)}(B) = \frac{C}{r} \left( 1 - \frac{1}{ \left( 1 + r \frac{\tau}{n} \right)^n } \right) - P^{(k)}(B)
\tag{12}
\]

Where \(P^{(k)}(B)\) is corresponds to the value of the option that complies with equation (11). In accordance with what was previously stated, \(n\), the number of time intervals, should be equal to 4.

In turn, equation (12) is valid for values that comply with the following restrictions:

\[
\lim_{B \to \infty} \nu^{(k)}(B) = \frac{C}{r} \left( 1 - \frac{1}{ \left( 1 + r \frac{\tau}{n} \right)^n } \right)
\tag{13}
\]

\[
\lim_{B \to B_k} \nu^{(k)}(B) = B_k
\tag{14}
\]

\(^{10}\) \(k\) is equal to the number of sub-periods in relation to the total number of time intervals, \(n=4\), considered in the model. In this way, the expression of the mortgage value for \(\frac{k\tau}{4}\) time periods is calculated with \(k = 1,2,3,4\).
\[ \lim_{B \to B_k} \frac{\partial V^{(k)}(B)}{\partial B} = 1 \]  \hspace{1cm} (15)

Condition (13) stipulates that, when the price of the property tends towards infinity, the value of the mortgage coincides with the present value of the future loan instalments. In this situation the value of the default option is zero. The delimitation of the lowest value for the mortgage price interval is given by equations (14) and (15). These restrictions impose the value of the mortgage contract to be equal to the price of the property, when this tends towards its critical level, \( B_k \). Consequently, for \( B \leq B_k \), the borrower does not fulfil his financial obligations and loses the property. In accordance with these conditions, the value of the mortgaged asset, before the maturity of the contract, should comply with the following relationship: \( V^{(k)}(B) \leq B \).

The solution of (12), subject to restrictions (13), (14), and (15) leads to a general solution for the mortgage value, which is given by the different expressions, in accordance with interval considered for the price of the property. The period to maturity is divided into \( n \) time intervals, \( n = 1,2,3,4 \). The formulae presented correspond to the value of the mortgage whose term until maturity is equal to \( \frac{k\tau}{n} \) with \( k = 1,2,3,4 \) and \( k \leq n \).

Therefore, for \( B \geq B_1 \), \( k = 1,2,3,4 \) and \( n = 1,2,3,4 \) the mortgage value is given by:

\[ \text{---} \]

\( B_1 \) corresponds to the critical value of the property over a period of possession of the contract equal to \( \frac{\tau}{n} \).
$$V_{B \geq B_i}^{(k)} (B) = \left( \frac{1}{1 + r \frac{\tau}{n}} \right)^n B - B \frac{1}{r} \left( 1 - \left( \frac{1 + r \frac{\tau}{n}}{1 + r} \right)^n \right) \frac{-2r \tau + 2r b + r \sigma^2}{2r \sigma^2}$$

\[
\left[ a_k + 2na_{k-1} \left( \frac{\tau \sigma^2}{\alpha_n} + \frac{\ln(B)}{\sqrt{\alpha_n}} \right) \right] \\
+ 2n^2 a_{k-2} \left( \frac{\ln(B)}{\sqrt{\alpha_n}} + \frac{2\tau \sigma^2}{\alpha_n} \right)^2 + \frac{n^4 a_{k-3}}{3} \left( \frac{30 \tau^3 \sigma^6}{\alpha_n^3} \right) \\
+ \frac{30 \tau^2 \sigma^4 \ln(B)}{\alpha_n^{3/2}} + \frac{9 \sigma^2 \tau \ln(B)^2}{\alpha_n^2} + \frac{\ln(B)^3}{\alpha_n^{3/2}} \right]
\]

Where:

$$a_{k-i} = 0 \text{ for } k - i \leq 0 \text{ and } I = 1,2,3.$$  

For $B \leq B_k^{12}$ and $k = 1,2,3,4$, the mortgage value becomes:

$$V_{B \leq B_k}^{(k)} (B) = B \text{ if } B \leq B_k$$  \hspace{1cm} (17)$$

In relation to the area where $B_2 \leq B \leq B_1$, for $k = 2,3,4$ and $n = 2,3,4$ the value is given by:

\[\text{Corresponds to the default range defined in the contract.}\]
Where:

\[ y_{k-i} = 0 \] and \[ z_{k-i} = 0 \] for \( k-i \leq 0 \) and \( i = 2,3 \)

In turn, for \( B_3 \leq B \leq B_2 \), \( k = 3,4 \) and \( n = 3,4 \), the mortgage value is given by:

\[
V_{(B_3 \leq B \leq B_2)}^{(k)}(B) = \frac{C \left( 1 - \frac{1}{\left( 1 + \frac{r}{n} \right)^n} \right) \left[ \left( 1 + \frac{\alpha}{n} \right)^{k-2} - 1 \right]}{r \left( 1 + \frac{\alpha}{n} \right)^{k-2}} - B \frac{-2r + 2 \beta + \alpha^2 + \sqrt{a_n}}{2 \sigma^2}
\]

\[
\left[ w_{k-2} + 2n w_{k-3} \left( \frac{\tau \sigma^2}{\alpha_n} - \frac{\ln(B)}{\sqrt{\alpha_n}} \right) \right] + \frac{B}{\left( 1 + b \frac{\alpha}{n} \right)^{k-2}}
\]

\[
- B \frac{-2r + 2 \beta + \alpha^2 + \sqrt{a_n}}{2 \sigma^2} \left( x_{k-2} + 2n x_{k-3} \left( \frac{\tau \sigma^2}{\alpha_n} + \frac{\ln(B)}{\sqrt{\alpha_n}} \right) \right)
\]

if \( B_3 \leq B \leq B_2 \)
Where:

\[ w_{k-i} = 0 \text{ and } x_{k-i} = 0 \text{ for } k - i \leq 0 \text{ and } i = 3 \]

For \( B_4 \leq B \leq B_3 \), \( k = 4 \) and \( n = 4 \), the mortgage value becomes:

\[
V_{B_4 \leq B \leq B_3}^{(k)}(B) = C \left( 1 - \frac{1}{\left( 1 + \frac{r}{n} \right)^n} \right) \left[ \left( 1 + \frac{r}{n} \right)^{-1} \right] + \frac{B}{r \left( 1 + \frac{r}{n} \right)} - \frac{1 + b}{n} \left( \frac{-2r + 2b + \sigma^2 + \sqrt{\alpha_n}}{2\sigma^2} \right) - u_1 B \left( \frac{1 + b}{2\sigma^2} \right) - \frac{-2r + 2b + \sigma^2 - \sqrt{\alpha_n}}{2\sigma^2} - v_1 B \left( \frac{-2r + 2b + \sigma^2 - \sqrt{\alpha_n}}{2\sigma^2} \right) \left( \frac{-2r + 2b + \sigma^2 - \sqrt{\alpha_n}}{2\sigma^2} \right) \text{ if } B_4 \leq B \leq B_3 \]

(20)

Regarding equations (16), (17), (18), (19), and (20):

\[ \alpha_n = \left( 2r \tau + \tau \sigma^2 \right)^2 + 4b \tau^2 \left( -2r + b + \sigma^2 \right) + 8n \sigma^2 \]

These different expressions for the mortgage value contain a set of parameters whose value is determined by the solution of a system of equations. It is necessary to solve a set of equations that must be verified by the different mortgage value expressions. Thus, we arrive at:

For \( k = 1 \), \( n = 1, 2, 3, 4 \) and \( B = B_1 \):

\[
V_{B \leq B_1}^{(1)}(B) = V_{B \leq B_1}^{(1)}(B) \]

(21)

\[
\frac{\partial V_{B \leq B_1}^{(1)}(B)}{\partial B} = 1
\]

For \( k = 2, 3, 4 \), \( n = 2, 3, 4 \) and \( B = B_k \):
\[ V_{[B \leq B_1]}^{(k)}(B) = V_{[B_1 \leq B \leq B_{k-1}]}^{(k)}(B) \]  \hspace{1cm} (22)

For \( k = 2,3,4 \), \( n = 2,3,4 \) and \( B = B_1 \) :

\[ \frac{\partial V_{[B \leq B_1]}^{(k)}(B)}{\partial B} = \frac{\partial V_{[B_1 \leq B \leq B_{k-1}]}^{(k)}(B)}{\partial B} \]  \hspace{1cm} (23)

For \( k = 3,4 \), and \( n = 3,4 \) \( B = B_2 \) :

\[ \frac{\partial V_{[B_1 \leq B \leq B_2]}^{(k)}(B)}{\partial B} = \frac{\partial V_{[B_2 \leq B \leq B_3]}^{(k)}(B)}{\partial B} \]  \hspace{1cm} (24)

For \( k = 4 \) and \( B = B_3 \) :

\[ \frac{\partial V_{[B_1 \leq B \leq B_3]}^{(k)}(B)}{\partial B} = \frac{\partial V_{[B_3 \leq B \leq B_4]}^{(k)}(B)}{\partial B} \]  \hspace{1cm} (25)

For a numerical determination of the critical values \( B_k \), for \( k = 2,3,4 \) and \( n = 2,3,4 \), a numerical solution for the following equation is required:

\[ \frac{\partial V_{[B_k \leq B \leq B_{k-1}]}^{(k)}(B)}{\partial B} = 1 \]  \hspace{1cm} (26)
2.2.4. Application of Richardson’s Extrapolation to the Commercial Mortgage Valuation Model

Application of Richardson’s extrapolation permits us to obtain a good estimate for the value of a function when continuous time is replaced by discrete time. Notwithstanding the fact that, in working out these formulae, the reference period is subdivided into a relatively small number of intervals - in this case four intervals - the results obtained are very close to those that would be obtained if time was to be considered as continuous.

Application of Richardson’s extrapolation to the mortgage valuation model results in a formula that allows us to calculate the extrapolated value of the mortgage, \( \hat{V}(0) \), and the critical price of the mortgaged property, \( \hat{B}_4(0) \).

The extrapolated value of the mortgage is given algebraically as follows:

\[
\hat{V}(0) = \frac{32}{3} \hat{V}(\frac{\tau}{4}) - \frac{27}{2} \hat{V}(\frac{\tau}{2}) + 4 \hat{V}(\frac{\tau}{4}) - \frac{1}{6} \hat{V}(\tau)
\]

(27)

Where the functions \( \hat{V}(\frac{\tau}{n}) \), for \( n = 1,2,3,4 \), correspond to the values obtained from the previously calculated functions, \( V^{(k)}(B) \), for \( n = k = 1,2,3,4 \).

In turn, the extrapolated critical value of the mortgaged property is given by:

\[
\hat{B}_4(0) = \frac{32}{3} \hat{B}_4(\frac{\tau}{4}) - \frac{27}{2} \hat{B}_4(\frac{\tau}{2}) + 4 \hat{B}_4(\frac{\tau}{4}) - \frac{1}{6} \hat{B}_4(\tau)
\]

(28)

Where \( \hat{B}_k(\frac{\tau}{n}) \) corresponds to the value of \( B_k \), for \( n = k = 1,2,3,4 \).

The numerical results obtained through the application of this model are presented in the following section, as well as analyses of mortgage price sensitivity in relation to the price of the mortgaged property. The aim of presenting these results is to demonstrate that
results obtained using the model developed in this study are consistent economic rationale.

3. Analysis of Results from the Mortgage Valuation Model

This section presents and discusses, both graphically and numerically, the results obtained using the general commercial mortgage valuation model previously developed. It does so by taking into account a basic set of economic parameters, using standard assumptions taken from the literature\(^\text{13}\). The analysis is carried out by way of graphs representing the evolution of the value of the mortgage asset, as well as tables presenting the numerical values for the critical price and the value of the mortgage.

Figures 1 to 4 illustrate mortgage values as a function of the underlying property value. In these figures, the time to maturity is 10 years and the annual instalment is €14,215. In figures 1 to 3, the volatility of the property price is 15% whereas, in figure 4, is 30%. Figures 1, 3 and 4, on one side, and figure 2, on the other side, take into consideration different levels of annual revenue – 5% and 10%, respectively. Regarding interest rates, figures 1, 2 and 3 consider 7.5% whereas figure 4, considers 2.5%.

In all calculations, the period to maturity was subdivided into four time intervals that encompass five distinct functions, in accordance with each specific property price interval.

The graphical representation of mortgage value allows for a general conclusion: whenever the property price is above the critical level, positive variations in the property price imply a growth in the value of the mortgage, this value being lower than that of the property. However, when the property price is below the critical level, the value of the mortgage coincides with the value of the property.
All the graphs show that the line representing mortgage value as a function of property price has a positive slope, being divided into distinct zones: a default zone, for property values below the critical price, where the value of the mortgage coincides with the value of the property, represented graphically by a straight line with a slope of 1; a zone of contract compliance, for property values above the critical price, which is represented in the graphs by an increasing function with a decreasing growth rate. In every case the solution evolves smoothly across the state space.

The differences between the figures originate in the different values assumed for the model’s parameters. A comparison between figures 1 and 2 shows that an increase in the annual rate of revenue induces a reduction in the critical price of the property, and consequently, a reduction in the value of the mortgage, for property prices above the critical price. The revenue is a dividend like feature whose increase will reduce the attraction of default. Ceteris paribus, with higher revenue, it will be necessary a greater plummet in house prices to give reason for default. In turn, comparison of Figures 1 and 3 shows that a fall in interest rates, whilst all other parameters remain unchanged, induces a reduction in the critical price of the property, and consequently, a reduction in the value of the mortgage, for property prices above the critical price. The value of the default option will increase leading to a situation where the price will need plummet more to justify immediate exercise. Regarding Figures 1 and 4, it can be observed that an increase in volatility of the property price leads to a reduction in the critical price of the property and in the value of the mortgage. A rise in the property price volatility will increase the value of the default option which is a negative component of the mortgage value14. Immediate exercise will be less appealing and consequently the critical value will fall.

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13 See, for example, works by Titman and Torous (1989) and Riddiough and Thompson (1993).
14 Vide expression (4).
Parameters underlying the construction of this figure are as follows: $n$, the number of time intervals is 4; $\tau$, the term remaining to maturity is 10 years; $r$, the interest rate is 0.075; $\sigma$, the volatility of the property price is 0.15; $b$, the annual rate of revenue is 0.05; $C$, the value of the fixed annual instalment is 14 215; and, $B_4$, the critical price of the property is 73 461.
Figure 2. Commercial Mortgage Value as a Function of $B$

Parameters underlying the construction of this figure are as follows: $n$, the number of time intervals is 4; $\tau$, the term remaining to maturity is 10 years; $r$, the interest rate is 0.075; $\sigma$, the volatility of the property price is 0.15; $b$, the annual rate of revenue is 0.10; $C$, the value of the fixed annual instalment is 14,215; and, $B_4$, the critical price of the property is 57,293.
Figure 3. Commercial Mortgage Value as a Function of $B$

Values of parameters underlying the construction of this figure are as follows: $n$, the number of time intervals is 4; $\tau$, the term remaining to maturity is 10 years; $r$, the interest rate is 0.025; $\sigma$, the volatility of the property price is 0.15; $b$, the annual rate of revenue is 0.05; $C$, the value of the fixed annual instalment is 14 215; and, $\bar{B}_4$, the critical price of the property is 49 710.
Figure 4. Commercial Mortgage Value as a Function of $B$

Values of parameters underlying the construction of this figure are as follows: $n$, the number of time intervals is 4; $\tau$, the term remaining to maturity is 10 years; $r$, the interest rate is 0.075; $\sigma$, the volatility of the property price is 0.30; $b$, the annual rate of revenue is 0.05; $C$, the value of the fixed annual instalment is 14 215; and $\hat{B}_4$, the critical price of the property is 51 709.

To end this section, two tables are presented with results for the critical price of the property and the value of the commercial mortgage obtained through application of the model proposed in this work. Columns (1), (2), (3), and (4) show the different values for the parameters of the model, whilst columns (5) and (6) show the extrapolated values of the critical price, $\hat{B}_4(0)$, and the value of the mortgage, $\hat{V}(0)$. All values were obtained through application of the model and then subjected to Richardson’s extrapolation.
The difference between the two tables lies in the time to maturity and in the value of each fixed annual instalment. The value of the annual instalment was determined in order that the present value of the whole stream of future instalments, in continuous time, would reach an amount of (approximately) € 100000. It is shown that, ceteris paribus, extending the time to maturity of the loan reduces the critical price and the value of the mortgage, since the value of the default option tends to rise with an increase in the corresponding maturity. On the other hand, when the volatility is raised, the critical price comes down, reducing the default range of the contract, since once again the value of the option will increase.
Table 1. Values for Critical Price and for the Value of a Commercial Mortgage Whose Revenue is Proportional to the Value of the Property, for $\tau = 10$

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The values for the remaining parameters in the table are as follows: $n$, the number of time intervals is 4; $C$, the fixed annual instalment is 14 215; and, $r$, the interest rate is 0.075.
Table 2. Values for Critical Price and for the Value of a Commercial Mortgage Whose Revenue is Proportional to the Value of the Property, for $\tau = 3$

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The values for the remaining parameters in the table are as follows: $n$, the number of time intervals is 4; $C$, the fixed annual instalment is 37 224; and, $r$, the interest rate is 0.075.
The objective of this section was to test the viability of the model proposed in this work. Graphic and numeric analysis have shown that the results obtained, namely those analysing sensitivity of mortgage value in relation to property value, are consistent with economic intuition and evolve smoothly along the state space.

4. Conclusion

The emphatic growth observed in this type of contracts during the last two decades justifies a greater academic effort in order to develop adequate valuation models. In the vast majority of cases, literature in this field has presented only numerical solutions. In order to obtain these numerical solutions, highly complex calculation techniques are required, which makes the process overly slow and expensive. The main objective of this work is to make a contribution in a hitherto under-explored area of financial research: the development of a contingent claims commercial mortgage valuation model with a closed form solution. Despite the fact that it does not take into account the prepayment option, and that it does not consider stochastic interest rates, the model developed in the present paper constitutes one of the first attempts to identify closed form solutions for commercial mortgage valuation. It is also a valid alternative to models proposed up to now in the specific field of commercial mortgage valuation. The corresponding results are much easier and quicker to find than the numerical solutions normally obtained in mortgage valuation models. Additionally, as can be inferred from the graphical representation of mortgage price sensitivity in relation to the value of the mortgaged property, these results make economic sense and evolve smoothly across the state space, evidencing the reasonability of the approach developed.

It is hoped that this study has contributed in some way towards advancing an area in which there has been little investigation, up to now. However, it is important to recognise that there is considerable room for further research. Following directly from our work, a significant improvement could result from the development of a general solution
applicable independently of the number of periods used in the discretization process. Another, more ambitious improvement, would be the development of a model whose final result was given by an analytic solution in continuous time. In this situation, it would not be necessary to apply Richardson’s extrapolation and the whole valuation procedure would become considerably more elegant.
BIBLIOGRAPHY


