UNDERDIFFERENCING VERSUS OVERDIFFERENCING TESTS

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Abstract: In the ARIMA modeling of business, ¯nancial, and macroeconomic time series, a very important question consists on the determination of the degree of integration of the data under analysis. There are essentially three di®erent approaches to this problem. In the ¯rst approach, the analyst tests for underdi®erencing, i.e., for the existence of one or more autoregressive unit roots, in order to decide whether further di®erencing is necessary. In the second approach, the analyst tests for overdi®erencing, i.e., for the existence of a unit root in the moving average polynomial of the ARMA model. In the third approach, the analyst directly estimates the degree of integration of the time series via spectral or time-domain methods. In this paper, we provide a Monte Carlo comparison of these three di®erent approaches. In particular, we contrast the power of the di®erent tests in fractionally integrated and in nearly nonstationary and nearly noninvertible processes.

1 Introduction

Consider the classical autoregressive integrated moving average models, ARIMA. We say that the random process x_t is an ARIMA$(p; d; q)$ if there are constant polynomials $Á(¢)$ and $µ(¢)$ in the backwards shift operator B, of order p and q, respectively, and a nonnegative integer d such that, for every t

$$Á^{¤}(B)x_t = Á(B)(1 ¡ B)^d x_t = µ(B)"_t \qquad (1)$$

where $"_t$ » iid$(0; ¾^2)$. As a regularity condition, we assume that the polynomials $Á(B)$ and $µ(B)$ have no

the degree of integration of the data under ana
or, in other words, the appropriate degree of d
encing to use. By appropriate degree of di®eren
we mean the (integer) order D such that the
di®erences of the time series x_t yield a statio
(causal) and invertible process. In the conte
ARIMA modeling, this integer is the paramete
(1), i.e., $D = d$.

A more general model which has recently rec
much attention is the fractionally integrated AR
or ARFIMA. In this model, d is allowed to be
real number and $(1 ¡ B)^d$ is de¯ned by its expa
$\sum \binom{d}{k} (¡ B)^k$. This model is nonstationary when
1=2 and noninvertible when d ¡1=2. See,
Brockwell and Davis (1991, section 13.2) for de

In the context of ARFIMA modeling the ap
priate degree of di®erencing can still be co
ered as the integer order D that renders the s
$(1 ¡ B)^D x_t$ a stationary (causal) and invertible
i.e., $D = [d + 1=2]$, where $[¢]$ represents the gre
integer function.[1]

In both ARIMA and ARFIMA modeling a
cial question is the determination of the approp
degree of integer di®erencing to use.

There are essentially three di®erent approach
this problem. In the ¯rst approach the analyst
for underdi®erencing, in the second approach th
alyst tests for overdi®erencing, and in the third
proach the analyst directly estimates the degr
integration.

The ¯rst approach, testing for an autoregre
unit root, has been the subject of much study
number of authors and includes a very large bo
literature. Common tests are, inter alia, the Di

(ADF), and the tests of Phillips and Perron (1988). See Diebold and Nerlove (1990) for a review.

The second approach, testing for a moving-average unit root, has recently began to receive more attention after the works of Tanaka (1990), Saikkonen and Luukkonen (1993), Tsay (1993), and Breitung (1994), among others.

The third approach, direct estimation of the degree of integration, has been suggested by Sowell (1992), who stressed that underdifferenced ARIMA processes and overdifferenced (noninvertible) ARMA processes can be nested as special cases of fractionally integrated models, and that the degree of integration d of the time series can be directly estimated by fitting an ARFIMA model to the series. Sowell proposed the exact maximum likelihood estimator for the ARFIMA model. This estimator, however, has a number of computational problems and becomes quickly intractable for a large number of observations. The use of a spectral method as in Geweke and Porter-Hudak (1983) or in Robinson (1993) has also been suggested as a tool for identification purposes by Crato (1992) and Hassler (1993).

In the presence of an ARFIMA process with $d \neq 0$ the determination of the appropriate degree of differencing is not an easy task. As Diebold and Rudebush (1991) demonstrated, the power of autoregressive unit root tests is low against ARFIMA models with $0 < d < 1/2$. Moreover, ARFIMA models with a positive d close to but less than 1/2 display a set of autocorrelation, spectral, and graphical properties that are almost indistinguishable from those of a nonstationary ARIMA (Crato, 1992). Consequently, the assumption of an ARIMA$(p, 1, q)$ in presence of a stationary ARFIMA is likely to occur. The problem at stake, then, is the detection of a unit root in the moving average part of the resulting overdifferenced series. We investigate, in particular, which type of test, or combination of tests, if any, is more powerful for these ARFIMA models: the AR unit root tests applied on the original near nonstationary ARFIMA(p, d, q) process or the MA unit root tests applied on the overdifferenced ARFIMA$(p, d-1, q)$ process.

2 The tests

Augmented Dickey-Fuller ADF(p_{ma}

The Augmented Dickey-Fuller (ADF) test of th
pothesis that $\phi^*(B)$ in (1) has a unit root is b
on the regression t-statistic for the hypothesis
$\alpha = 1$ in

$$x_t = \alpha x_{t-1} + \pi_1 \Delta x_{t-1} + \pi_2 \Delta x_{t-2} + \cdots \pi_p \Delta x$$

where $\Delta = (1 - B)$. The selection of the tru
tion parameter p used in our study follows a gen
to-specific adaptive rule. We first choose a an
per bound parameter p_{max} and test the hyp
sis that the coefficient π_{max} is significantly di
from zero. If it is, we set $\hat{p} = p_{max}$ and com
the Dickey-Fuller statistic based on this autore
sion. Otherwise, we run an autoregression o
der $p_{max} - 1$ and test the hypothesis that $\pi_{p_{max}}$
zero. The selection procedure stops when the
significant coefficient is found, getting the sta
ADF(p_{max}). Hall (1994) and Ng and Perron (1
provide simulation studies showing that this p
dure tends to minimize the size distortions w
reported for the ADF test.

In contrast to the ADF test, the following
cedures test for overdifferencing. We closely f
the notation of Tanaka (1990) and Breitung (1
The time series under consideration will alwa
represented by y_t, $t = 1, \ldots, T$ and the integ
time series, possibly nonstationary but suppo
invertible, will be represented by z_t, $t = 1, \ldots$
If the original observations were differenced yie
the series y_t then the integrated series will b
served. Hereafter, we will suppose that this i
the case and so z_t will be constructed. To this
we will use the Helmert transform

$$z_t = Hy_t = (t + t^2)^{-1/2} \sum_{j=1} j y_j .$$

We do not use the partial sums of y_t since, u
the null of a moving-average unit root, they w
constitute a sequence with all values depending
stantially on a common initial factor.

Tanaka's statistic τ_T

The Tanaka (1990) statistic compares the varia in probability to a fixed quantity while the sample variance of Hz_t, normalized by T, will converge to a functional of the Brownian motion. The statistic can be written as

$$\tau_T = \frac{\sum_{t=1}^{T}(Hz_t)^2}{T\sum_{t=1}^{T} z_t^2} \qquad (3)$$

and its limiting distribution, as well as its finite-sample and limiting power against invertible MA(1) alternatives were tabulated by Tanaka (1990, p. 438).

Corrected Tanaka's statistic $\hat{\tau}_T$

In order to allow for MA models of higher order, Tanaka (1990) suggested a non-parametric correction to the variance estimator. Using a Bartlett window up to lag l for estimating the spectral density f_z he gets

$$\hat{\tau}_T(l) = \frac{\sum_{t=1}^{T}(Hz_t)^2}{T^2 2\pi \hat{f}_z(0)}. \qquad (4)$$

Breitung's correction $\tilde{\tau}_T$

Breitung (1994) suggested the use of another spectral estimator: $2\pi\tilde{f}_z(0) = \sum_{j=-q}^{q}\tilde{\gamma}_z(j)$ where $\tilde{\gamma}_z(j)$; $j = 1, \ldots, q$, are obtained from the sample autocovariances of z_t, $\hat{\gamma}_z(j) = \hat{\gamma}_z(-j)$, as follows

$$\tilde{\gamma}_z(j) = \frac{1}{2}\sum_{k=-q}^{q}(q+1-\max\{j, |k|\})\,\hat{\gamma}_z(k). \qquad (5)$$

Under the null of an MA(q) model with a simple unit root, $\tilde{\gamma}_z(j)$ are the autocovariance estimates of the process z_t. See Breitung (1994, Lemma 1) for details. The statistic becomes

$$\tilde{\tau}_T(q) = \frac{\sum_{t=1}^{T}(Hz_t)^2}{T^2 2\pi \tilde{f}_z(0)}. \qquad (6)$$

Breitung's spectral statistic ϕ_T

This spectral statistic assumes an MA(q + 1) null, with a simple unit root, and tests the hypothesis $f_y(0) = 0$. Under this condition, the series of crossproducts ξ_t; $t = q+1, \ldots, T$, is constructed

$$\xi_t = z_t^2 + 2\sum_{j=1}^{q} z_t z_{t-j}.$$

A consistent estimator for $f_\xi(0)$ is

and a consistent estimator for the correspon
variance is

$$\hat{V}_\xi = \sum_{j=-q}^{q}\left(1 - \frac{|j|}{q+1}\right)\hat{\gamma}_\xi(j);$$

with $\hat{\gamma}_\xi(j) = \hat{\gamma}_\xi(-j) = T^{-1}\sum_t \xi_t\xi_{t-j}$. The resu
statistic

$$\phi_T(q) = \frac{T\bar{\xi}^2}{\hat{V}_\xi}$$

is asymptotically χ^2 distributed under the nul
allows a (two-sided) test of the hypothesis $f_y(0)$

Breitung's variance difference test

This test compares two estimators for the varian
the integrated series z_t. One estimator, S_1^2, is si
the sample variance $T^{-1}\sum z_t^2$. The other estim
S_0^2, is constructed under the null of a unit root
(5),

$$S_0^2 = \frac{1}{2}\sum_{k=-q}^{q}(q+1-|k|)\,\hat{\gamma}_z(k).$$

The normalizing estimator $\tilde{V}_T$ is obtained fro
sample autocorrelations for the integrated seri

$$\tilde{V}_T = \sum_{j=-q+1}^{q-1}\rho_{Hz}(j).$$

Under the null of a MA(q) process with a simple
root, the statistic

$$\lambda_T(q) = \frac{\sqrt{T}(S_1^2 - S_0^2)}{S_0^2\sqrt{\tilde{V}_T}}$$

is asymptotically standard normal.

All previous procedures test either for und
ferencing (ADF) or for overdifferencing (Tana
and Breitung ϕ and λ). In contrast, the spectr
gression procedure we now describe can be use
both purposes.

Spectral regression tests $\hat{d}$

Geweke and Porter-Hudak (1983) suggested a
timation method for the fractional integratio

function of the frequencies. For low-order frequencies, the residuals e_j of the following regression equation approximate a white noise series

$$\ln I_\tau(j) = a + d \ln \left(\frac{1}{4} \sin^{-2} \frac{\pi j}{T} \right) + e_j. \qquad (9)$$

For the regression we use $j = m_L, \ldots, m_U$, where $m_L = [(T/10)^{1/3}]$ and $m_U = [T^{1/2}]$. For details see Geweke and Porter-Hudak (1983) and Robinson (1993). In presence of ARFIMA processes, tests for (I) underdifferencing and (III) overdifferencing can be formalized as follows

$$\text{(I)} \quad \left\{ \begin{array}{ll} H_0: & d \le 1/2 \\ H_1: & d > 1/2 \end{array} \right. \qquad \text{(III)} \quad \left\{ \begin{array}{ll} H_0: & d \ge -1/2 \\ H_1: & d < -1/2 \end{array} \right.$$

In presence of ARIMA models, this test can still provide some useful information. First, the existence of a unit root in the MA polynomial implies a root on the spectrum of the process at frequency zero. The spectral regression method should be able to detect such a root. Second, the existence of a unit root in the AR polynomial implies a divergence of the periodogram at frequency zero. Although the spectrum of such nonstationary process is not defined, Crato (1992) proved that the spectral regression estimate still converges to the value of d, i.e., 1. Hurvich and Ray (1995) confirmed the asymptotic unbiasedness of this estimator for $d = 1$ and obtained general results for any $d < 3/2$. Considering only integer d, tests for (II) underdifferencing and (IV) overdifferencing can be formalized as follows.

$$\text{(II)} \quad \left\{ \begin{array}{ll} H_0: & d = 0 \\ H_1: & d > 0 \end{array} \right. \qquad \text{(IV)} \quad \left\{ \begin{array}{ll} H_0: & d = 0 \\ H_1: & d < 0 \end{array} \right.$$

We perform these (I)-(IV) tests with the t ratios given by the standard deviations of the regressions.

3 Results

Our simulations include the simple fractional noise model ARFIMA$(0, d, 0)$, two nearly nonstationary AR(1) processes, and a pair of comparable nonstationary and nearly nonstationary models presented by Wichern (1973).

The ARFIMA(0,d,0) models have the following

-0.1, -0.2, -0.45, -0.55, -0.8, and -1.0. The
$d = 1.0$, 0.0, and -1.0 are, respectively, the si
ARIMA(0,1,0), the white noise, and the ARIM
1,0).

The two AR(1) models we generated have ϕ
and $\phi = 0.95$.

The two Wichern models are the ARIMA((
$z_t - z_{t-1} = \varepsilon_t - 0.8\varepsilon_{t-1}$ and the ARMA
$z_t - 0.95 z_{t-1} = \varepsilon_t - 0.74\varepsilon_{t-1}$. Wichern (1973)
structed these two models in order to illustrat
possibility of having pairs of nonstationary and
tionary processes with almost undistinguishabl
ond order sample properties for a given size T
this case $T = 100$.

All computations were done using the Gauss
gramming language and Gauss built-in funct
The fractionally integrated series were generate
multiplying the Choleski decomposition of the
covariance matrix by a vector of independent
dard normal random variables. For each case
replications were performed.

The ADF tests were performed over the ori
series, the MA unit root tests were performed
the differenced series, and the spectral regre
were performed over both the original (I, II)
the differenced series (III, IV).

For all but the Wichern models we used T
100, 250, and 500.

Tables 1 through 9 report the fraction of rejec
of the corresponding null hypothesis. The level o
tests was set to 5%.

4 Conclusions

The results from the simulated ARFIMA model
bles 1 to 4) show the insufficiencies of all st
tests. While the AR unit root test has diff
discriminating between nonstationary and sta
ary ARFIMA, the MA unit root tests have diff
discriminating between noninvertible and inve
stationary ARFIMA.

No MA unit root test dominates uniforml
others. Breitung's correction to Tanaka's stat
$\tau_T(q)$, and the variance difference statistic, λ
seem to perform better. However, they both
critically dependent on the choice of the trunc
parameter q

For a reasonable sample size, say $T = 250$, the ADF test is generally able to reject the AR unit root of a stationary ARFIMA$(0; d; 0)$, $(d < 1=2)$. Meanwhile, for the same sample size, the MA unit root tests have trouble detecting the noninvertibility of the corresponding di®erenced series, the ARFIMA$(0; d ¡ 1; 0)$.

This seems to indicate that little can be gained from the use of the MA unit root tests only. Tsay had suggested the construction of a nonstationary series from a stationary but possibly noninvertible one in order to apply the well-known AR unit root tests. This avoids the use of the MA unit root tests alone for testing the overdi®erencing of a time series.

However, testing for overdi®erencing may be a sensible complement to the underdi®erencing tests. In particular, the spectral ($Á_T$) and variance-di®erence ($¸_T$) tests seem to have an appreciable shift in power when the fractionally integration parameter of the ARFIMA model moves from the nonstationary to the stationary range.

In the case of the AR(1) models (tables 5 to 8), we notice that a choice of a test is equally di±cult. The MA unit root tests seem to perform better than the ADF test in some particular circumstances. The spectral ($Á_T$) and variance-di®erence ($¸_T$) tests outperform the various variants of Tanaka's test. However, they are all very sensitive to the (somewhat arbitrary) choice of the truncation parameter q and so any conclusion is hard to establish.

In the two Wichern models (table 9), all tests perform very poorly. However, with the exception of the spectral regression tests, both the ADF and the MA tests give some indication of detecting more often nonstationarity for the ARIMA model than for the ARMA model. The $¿_T$, in particular, reveals a better capability of distinguishing between the two cases. This may be an additional indication that the use of MA unit root tests may be an useful additional tool.

References

Breitung, JÄrg (1994). \Some simple tests of the moving-average unit root hypothesis," Journal of Time Series Analysis 15, 251{270.

Crato, Nuno (1992). \Periodogram analysis of nc tionary random variables," Department of Math ical Sciences, University of Delaware.

Crato, Nuno (1992). \Long-memory time series mis i¯ed as nonstationary ARIMA," American Stat Association, Business and Economic Statistics S Proceedings, 82{87.

Dickey, David A. and Fuller, Wayne (1979). \D bution of the estimates for autoregressive time with a unit root," Journal of the American Stat Association 74, 427{431.

Diebold, Francis X. and Nerlove, Marc (1990). roots in economic time series: A selective survey, vances in Econometrics, vol. 8, 3{69.

Diebold, Francis X. and Rudebush, Glenn D. (1 \On the power of Dickey-Fuller tests against frac alternatives," Economic Letters 35, 155{160.

Geweke, John and Porter-Hudak, Susan (1983). estimation and application of long memory time models," Journal of Time Series Analysis 4, 221

Hall, Alastair (1994). \Testing for a unit root in tir ries with pretest data-based model selection," Jo of Business and Economic Statistics 12, 461{47

Hurvich, Cli®ord and Ray, Bonnie K. (1995). \Es tion of the memory parameter for nonstationa noninvertible fractionally integrated processes," nal of Time Series Analysis 16, 17{41.

Kwiatkowski, Denis, Phillips, Peter C.B., Schmid ter, and Shin, Yongcheol (1992). \Testing th hypothesis of stationarity against the alternativ unit root: How sure are we that economic tin ries have a unit root?," Journal of Econometric 159{178.

Lee, Dongin and Schmidt, Peter (1993). \On the I of the KPSS test of stationarity against fractic integrated alternatives," Econometrics and Eco Theory Paper No. 9111, Michigan State Univers

Lo, Andrew W. (1991). \Long-term memory in market prices," Econometrica 59, 1279{1313.

Ng, Serena and Perron, Pierre (1995). \Unit root in ARMA models with data-dependent metho teh selection of the truncation lag," Journal c American Statistical Association 90, 268{281.

Phillips, Peter C.B. and Perron, Pierre (1988). \T for a unit root in time series regression," Biom 75, 335{346.

Robinson, Peter M. (1993). \Log-periodogram r sion of time series with long range dependence," don School of Economics.

Saikkonen, Pentti and Luukkonen, Ritva (1993). \ ing for a moving average unit root in autoregr integrated moving average models," Journal c American Statistical Association 88, 596{601.

Sowell, Fallaw (1992). \Modeling long-run behavio

Tsay, Ruey S. (1993). \Testing for noninvertible models with applications," Journal of Business and Economic Statistics 11, 225{233.

Wichern, Dean W. (1973). \The behaviour of the sample autocorrelation function for an integrated moving-average process," Biometrika 60, 235{239.

Table 1: Simulated ARFIMA(0,d,0) Models { T = 50

Tests	d=1.0	d=0.9	d=0.7	d=0.6	d=0.55	d=0.45	d=0.3	d=0
ADF(8)	0.061	0.112	0.202	0.348	0.387	0.538	0.953	0.999
$\hat{d}$ (I)	0.299	0.260	0.177	0.121	0.099	0.071	0.038	0.024
$\hat{d}$ (II)	0.607	0.571	0.473	0.363	0.301	0.255	0.201	0.082
$\hat{d}$ (III)	0.021	0.034	0.042	0.079	0.077	0.090	0.121	0.195
$\hat{d}$ (IV)	0.093	0.126	0.163	0.252	0.252	0.296	0.357	0.454
¿T	0.950	0.920	0.861	0.755	0.696	0.569	0.341	0.046
¿̃T (1)	0.849	0.797	0.720	0.617	0.554	0.454	0.270	0.046
¿̃T (4)	0.630	0.570	0.486	0.410	0.358	0.275	0.153	0.029
¿̃T (8)	0.480	0.439	0.330	0.274	0.225	0.164	0.080	0.009
¿̃T (12)	0.400	0.331	0.227	0.175	0.127	0.080	0.029	0.000
¿̇T (1)	0.973	0.952	0.889	0.758	0.712	0.566	0.361	0.107
¿̇T (4)	0.758	0.701	0.569	0.467	0.411	0.321	0.193	0.079
¿̇T (8)	0.493	0.439	0.333	0.264	0.210	0.175	0.101	0.037
¿̇T (12)	0.362	0.295	0.212	0.174	0.131	0.104	0.074	0.019
ÁT (1)	0.891	0.795	0.598	0.355	0.300	0.203	0.105	0.031
ÁT (4)	0.198	0.150	0.068	0.033	0.037	0.031	0.015	0.027
ÁT (8)	0.019	0.022	0.008	0.012	0.011	0.016	0.013	0.021
ÁT (12)	0.003	0.002	0.004	0.005	0.003	0.000	0.009	0.010
¸T (1)	0.993	0.975	0.888	0.666	0.604	0.346	0.090	0.007
¸T (4)	0.423	0.313	0.141	0.045	0.031	0.008	0.004	0.000
¸T (8)	0.042	0.018	0.002	0.000	0.000	0.000	0.000	0.001
¸T (12)	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

Table indicates the fraction of rejections of the null hypothesis of each test

Table 2: Simulated ARFIMA(0,d,0) Models { T = 100

Tests	d=1.0	d=0.9	d=0.7	d=0.6	d=0.55	d=0.45	d=0.3	d=0
ADF(10)	0.060	0.136	0.297	0.569	0.676	0.741	0.999	1.000
$\hat{d}$ (I)	0.331	0.267	0.165	0.092	0.090	0.060	0.025	0.012
$\hat{d}$ (II)	0.666	0.580	0.463	0.351	0.312	0.228	0.169	0.079
$\hat{d}$ (III)	0.018	0.026	0.028	0.055	0.070	0.087	0.115	0.214
$\hat{d}$ (IV)	0.077	0.105	0.137	0.220	0.250	0.284	0.331	0.475
¿T	0.994	0.980	0.957	0.900	0.868	0.738	0.526	0.052
¿̃T (1)	0.955	0.932	0.866	0.779	0.718	0.597	0.408	0.056
¿̃T (4)	0.762	0.741	0.644	0.540	0.465	0.374	0.250	0.040
¿̃T (8)	0.628	0.616	0.517	0.399	0.337	0.280	0.161	0.036
¿̃T (12)	0.539	0.528	0.430	0.308	0.254	0.194	0.109	0.024
¿̇T (1)	1.000	0.992	0.977	0.911	0.882	0.741	0.530	0.099
¿̇T (4)	0.945	0.906	0.810	0.684	0.609	0.491	0.325	0.097
¿̇T (8)	0.794	0.735	0.609	0.480	0.405	0.303	0.200	0.053
¿̇T (12)	0.632	0.590	0.468	0.349	0.296	0.220	0.138	0.032
ÁT (1)	0.998	0.991	0.910	0.672	0.557	0.394	0.178	0.030
ÁT (4)	0.675	0.464	0.200	0.106	0.065	0.043	0.026	0.031
ÁT (8)	0.264	0.143	0.066	0.024	0.034	0.022	0.034	0.028
ÁT (12)	0.098	0.071	0.034	0.025	0.023	0.026	0.030	0.025
¸T (1)	1.000	1.000	0.999	0.960	0.916	0.722	0.266	0.001
¸T (4)	0.933	0.834	0.642	0.351	0.213	0.101	0.015	0.000
¸T (8)	0.505	0.358	0.152	0.046	0.018	0.002	0.001	0.000
¸T (12)	0.203	0.122	0.027	0.003	0.001	0.000	0.000	0.000

Table 3: Simulated ARFIMA(0,d,0) Models { T = 250

Tests	d=1.0	d=0.9	d=0.7	d=0.6	d=0.55	d=0.45	d=0.3	d=0
ADF(12)	0.052	0.158	0.450	0.794	0.873	0.959	1.000	1.000
$\hat{d}$ (I)	0.518	0.384	0.223	0.113	0.080	0.044	0.022	0.000
$\hat{d}$ (II)	0.903	0.844	0.712	0.557	0.509	0.398	0.256	0.061
$\hat{d}$ (III)	0.003	0.009	0.021	0.041	0.040	0.082	0.137	0.283
$\hat{d}$ (IV)	0.061	0.109	0.216	0.330	0.340	0.437	0.565	0.718
¿T	1.000	1.000	0.996	0.988	0.974	0.944	0.707	0.050
¿̂T (1)	0.998	0.990	0.984	0.949	0.925	0.851	0.597	0.050
¿̂T (4)	0.953	0.930	0.877	0.786	0.752	0.646	0.408	0.052
¿̂T (8)	0.864	0.802	0.723	0.643	0.571	0.496	0.308	0.045
¿̂T (12)	0.781	0.705	0.612	0.542	0.476	0.412	0.254	0.045
¿̇T (1)	1.000	1.000	0.999	0.996	0.985	0.950	0.703	0.063
¿̇T (4)	0.999	0.992	0.977	0.925	0.898	0.790	0.511	0.110
¿̇T (8)	0.988	0.959	0.905	0.786	0.735	0.627	0.372	0.096
¿̇T (12)	0.950	0.890	0.801	0.676	0.612	0.511	0.289	0.064
ÁT (1)	1.000	1.000	1.000	0.983	0.938	0.806	0.392	0.023
ÁT (4)	0.997	0.939	0.656	0.258	0.211	0.102	0.046	0.034
ÁT (8)	0.856	0.631	0.263	0.092	0.064	0.040	0.044	0.036
ÁT (12)	0.610	0.368	0.135	0.046	0.045	0.042	0.026	0.040
¸T (1)	1.000	1.000	1.000	1.000	1.000	0.996	0.722	0.001
¸T (4)	1.000	1.000	0.995	0.933	0.849	0.594	0.129	0.000
¸T (8)	0.987	0.967	0.852	0.516	0.384	0.172	0.019	0.000
¸T (12)	0.917	0.806	0.555	0.218	0.145	0.052	0.002	0.000

Table indicates the fraction of rejections of the null hypothesis of each test

Table 4: Simulated ARFIMA(0,d,0) Models { T = 500

Tests	d=1.0	d=0.9	d=0.7	d=0.6	d=0.55	d=0.45	d=0.3	d=0
ADF(14)	0.068	0.205	0.591	0.942	0.964	0.998	1.000	1.000
$\hat{d}$ (I)	0.593	0.488	0.296	0.105	0.081	0.041	0.007	0.000
$\hat{d}$ (II)	0.954	0.913	0.822	0.710	0.636	0.508	0.300	0.056
$\hat{d}$ (III)	0.002	0.002	0.011	0.030	0.033	0.077	0.144	0.311
$\hat{d}$ (IV)	0.076	0.120	0.241	0.353	0.445	0.553	0.699	0.812
¿T	1.000	1.000	1.000	0.997	0.998	0.970	0.829	0.051
¿̂T (1)	1.000	0.999	0.997	0.988	0.976	0.928	0.727	0.050
¿̂T (4)	0.991	0.987	0.963	0.916	0.887	0.788	0.549	0.046
¿̂T (8)	0.963	0.942	0.879	0.799	0.778	0.637	0.424	0.042
¿̂T (12)	0.913	0.895	0.810	0.712	0.679	0.550	0.342	0.042
¿̇T (1)	1.000	1.000	1.000	1.000	0.999	0.982	0.834	0.059
¿̇T (4)	1.000	1.000	0.999	0.983	0.977	0.901	0.645	0.092
¿̇T (8)	0.998	0.998	0.978	0.932	0.896	0.769	0.502	0.098
¿̇T (12)	0.994	0.984	0.951	0.853	0.827	0.664	0.412	0.083
ÁT (1)	1.000	1.000	1.000	0.999	1.000	0.978	0.681	0.033
ÁT (4)	1.000	1.000	0.956	0.542	0.352	0.156	0.068	0.045
ÁT (8)	0.998	0.950	0.546	0.189	0.121	0.068	0.043	0.039
ÁT (12)	0.957	0.821	0.336	0.097	0.074	0.053	0.042	0.048
¸T (1)	1.000	1.000	1.000	1.000	1.000	1.000	0.979	0.002
¸T (4)	1.000	1.000	1.000	1.000	0.999	0.952	0.366	0.000
¸T (8)	1.000	1.000	0.998	0.942	0.879	0.567	0.101	0.000
¸T (12)	0.998	0.992	0.965	0.759	0.640	0.275	0.037	0.000

Table 5: Simulated AR(1) Models { T = 50

Tests	Á=0.8	Á=0.9	Á=0.95
ADF(8)	0.721	0.353	0.193
$\hat{d}$ (I)	0.143	0.164	0.205
$\hat{d}$ (II)	0.374	0.429	0.486
$\hat{d}$ (III)	0.022	0.023	0.021
$\hat{d}$ (IV)	0.109	0.082	0.103
¿$_T$	0.755	0.873	0.901
¿̂$_T$ (1)	0.517	0.688	0.763
¿̂$_T$ (4)	0.286	0.408	0.504
¿̂$_T$ (8)	0.151	0.266	0.340
¿̂$_T$ (12)	0.084	0.178	0.249
¿̇$_T$ (1)	0.774	0.917	0.930
¿̇$_T$ (4)	0.319	0.499	0.627
¿̇$_T$ (8)	0.146	0.251	0.351
¿̇$_T$ (12)	0.082	0.153	0.217
Á$_T$ (1)	0.792	0.837	0.869
Á$_T$ (4)	0.083	0.115	0.153
Á$_T$ (8)	0.003	0.012	0.014
Á$_T$ (12)	0.004	0.003	0.001
¸$_T$ (1)	0.856	0.956	0.968
¸$_T$ (4)	0.031	0.129	0.268
¸$_T$ (8)	0.000	0.003	0.007
¸$_T$ (12)	0.000	0.000	0.000

Table indicates the fraction of rejections of the null hypothesis of each test

Table 6: Simulated AR(1) Models { T = 100

Tests	Á=0.8	Á=0.9	Á=0.95
ADF(10)	0.985	0.736	0.383
$\hat{d}$ (I)	0.075	0.138	0.179
$\hat{d}$ (II)	0.271	0.418	0.480
$\hat{d}$ (III)	0.026	0.010	0.009
$\hat{d}$ (IV)	0.139	0.069	0.063
¿$_T$	0.803	0.943	0.968
¿̂$_T$ (1)	0.563	0.778	0.893
¿̂$_T$ (4)	0.264	0.466	0.600
¿̂$_T$ (8)	0.154	0.284	0.429
¿̂$_T$ (12)	0.109	0.212	0.361
¿̇$_T$ (1)	0.851	0.980	0.994
¿̇$_T$ (4)	0.354	0.671	0.834
¿̇$_T$ (8)	0.153	0.355	0.574
¿̇$_T$ (12)	0.100	0.220	0.374
Á$_T$ (1)	0.990	0.997	1.000
Á$_T$ (4)	0.227	0.453	0.563
Á$_T$ (8)	0.033	0.094	0.168
Á$_T$ (12)	0.017	0.024	0.033
¸$_T$ (1)	0.993	1.000	0.999
¸$_T$ (4)	0.121	0.541	0.779
¸$_T$ (8)	0.006	0.052	0.212
¸$_T$ (12)	0.001	0.002	0.032

Table 7: Simulated AR(1) Models { T = 250

Tests	Á=0.8	Á=0.9	Á=0.95
ADF(12)	1.000	1.000	0.907
$\hat{d}$ (I)	0.017	0.095	0.236
$\hat{d}$ (II)	0.253	0.510	0.685
$\hat{d}$ (III)	0.058	0.008	0.004
$\hat{d}$ (IV)	0.384	0.162	0.075
¿$_T$	0.844	0.972	0.996
¿̂$_T$ (1)	0.583	0.858	0.961
¿̂$_T$ (4)	0.262	0.514	0.741
¿̂$_T$ (8)	0.161	0.315	0.524
¿̂$_T$ (12)	0.121	0.234	0.389
¿̇$_T$ (1)	0.923	0.996	1.000
¿̇$_T$ (4)	0.375	0.799	0.964
¿̇$_T$ (8)	0.159	0.429	0.762
¿̇$_T$ (12)	0.104	0.270	0.565
Á$_T$ (1)	1.000	1.000	1.000
Á$_T$ (4)	0.686	0.940	0.972
Á$_T$ (8)	0.116	0.369	0.621
Á$_T$ (12)	0.038	0.128	0.287
¸$_T$ (1)	1.000	1.000	1.000
¸$_T$ (4)	0.444	0.977	0.996
¸$_T$ (8)	0.013	0.274	0.789
¸$_T$ (12)	0.000	0.041	0.295

Table indicates the fraction of rejections of the nu pothesis of each test

Table 8: Simulated AR(1) Models { T = 500

Tests	Á=0.8	Á=0.9	Á=0.95
ADF(14)	1.000	1.000	1.000
$\hat{d}$ (I)	0.002	0.039	0.232
$\hat{d}$ (II)	0.134	0.529	0.817
$\hat{d}$ (III)	0.134	0.020	0.002
$\hat{d}$ (IV)	0.680	0.284	0.084
¿$_T$	0.849	0.984	1.000
¿̂$_T$ (1)	0.609	0.892	0.983
¿̂$_T$ (4)	0.278	0.552	0.793
¿̂$_T$ (8)	0.148	0.341	0.546
¿̂$_T$ (12)	0.101	0.258	0.394
¿̇$_T$ (1)	0.933	0.999	1.000
¿̇$_T$ (4)	0.385	0.868	0.991
¿̇$_T$ (8)	0.158	0.497	0.857
¿̇$_T$ (12)	0.099	0.300	0.616
Á$_T$ (1)	1.000	1.000	1.000
Á$_T$ (4)	0.956	1.000	1.000
Á$_T$ (8)	0.229	0.770	0.947
Á$_T$ (12)	0.061	0.312	0.700
¸$_T$ (1)	1.000	1.000	1.000
¸$_T$ (4)	0.822	1.000	1.000
¸$_T$ (8)	0.035	0.680	0.992
¸$_T$ (12)	0.007	0.138	0.732

Table 9: Simulated Wichern Models { T = 100

Tests	ARMA(1,1)	ARIMA(0,1,1)
ADF(10)	0.998	0.677
$\hat{d}$ (I)	0.055	0.052
$\hat{d}$ (II)	0.241	0.275
$\hat{d}$ (III)	0.110	0.126
$\hat{d}$ (IV)	0.305	0.305
$¿_T$	0.710	0.851
$\hat{¿}_T$ (1)	0.635	0.804
$\hat{¿}_T$ (4)	0.431	0.670
$\hat{¿}_T$ (8)	0.287	0.549
$\hat{¿}_T$ (12)	0.198	0.456
$\dot{¿}_T$ (1)	0.749	0.880
$\dot{¿}_T$ (4)	0.561	0.739
$\dot{¿}_T$ (8)	0.369	0.612
$\dot{¿}_T$ (12)	0.217	0.497
$Á_T$ (1)	0.034	0.031
$Á_T$ (4)	0.039	0.035
$Á_T$ (8)	0.042	0.035
$Á_T$ (12)	0.022	0.029
$¸_T$ (1)	0.458	0.599
$¸_T$ (4)	0.143	0.298
$¸_T$ (8)	0.010	0.035
$¸_T$ (12)	0.000	0.002

Table indicates the fraction of rejections of the null hypothesis of each test