Introduction to Dynamic Programming
Applied to Economics

Paulo Brito
Instituto Superior de Economia e Gestão
Universidade Técnica de Lisboa
pbrito@iseg.utl.pt

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Chapter 1

Introduction

We will study the three workhorses of modern macro and financial economics, using dynamic programming methods:

- the cake-eating (or resource depletion) problem,

- the intertemporal allocation problem for the representative agent in a finance economy,

- the Ramsey model,

in four different environments:

- discrete time and continuous time;

- deterministic and stochastic methodology

- we the dynamic programming approach,

- some heuristic proofs,

- and derive explicit equations whenever possible.
1.1 A general overview

We will consider the following types of problems:

1.1.1 Discrete time deterministic models

In the space of the sequences \( \{u_t, x_t\}_{t=0}^{\infty} \), such that \((u_t, x_t) \in \mathbb{R}^m \times \mathbb{R}\), and \( t \mapsto (u_t, x_t) \) are deterministic processes, that verify the sequence of budget constraints

\[
x_{t+1} = g(x_t, u_t), \quad t = 0, ..., \infty
\]
\[
x_0 \quad \text{given}
\]

where \( 0 < \beta \equiv \frac{1}{1+\rho} < 1 \), where \( \rho > 0 \), and in some cases, verifying

\[
\lim_{t \to \infty} \beta^{-t} x_t \geq 0,
\]

choose an optimal sequence \( \{u_t^*, x_t^*\}_{t=0}^{\infty} \) that maximizes the sum

\[
\max_{\{u\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t f(u_t, x_t)
\]

By applying the principle of dynamic programming the first order necessary conditions for this problem are given by the Hamilton-Jacobi-Bellman (HJB) equation,

\[
V(x_t) = \max_{u_t} \{f(u_t, x_t) + \beta V(x_{t+1})\}
\]

which is usually written as

\[
V(x) = \max_{u} \{f(u, x) + \beta V(g(u, x))\} \tag{1.1}
\]

If an optimal control \( u^* \) exists, it has the form \( u^* = h(x) \), where \( h(x) \) is called the policy function. If we substitute back in the HJB equation, we get a functional equation

\[
V(x) = f(h(x), x) + \beta V(g(h(x), x)].
\]
Then solving the HJB equation means finding the function $V(x)$ which solves the functional equation. If we are able to determine $V(x)$ (explicitly or numerically) then we can also determine $u^*_t = h(x_t)$. If we substitute in the difference equation, $x_{t+1}^* = g(x_t^*, h(x_t^*))$, starting at $x_0$, the solution $\{u_t^*, x_t^*\}_{t=0}^\infty$ of the optimal control problem can be determined, explicitly or implicitly.

Only in very rare cases we can find $V(.)$ explicitly.
1.1.2 Continuous time deterministic models

In the space of continuous functions of time \((u(t), x(t))\) choose an optimal flow \([u^*(t), x^*(t)]_{t \in \mathbb{R}^+}\) such that \([u^*(t)]_{t \in \mathbb{R}^+}\) maximizes the functional

\[
V[u] = \int_0^\infty f(u(t), x(t)) e^{-\rho t} dt
\]

where \(\rho > 0\), subject to the instantaneous budget and the initial state constraints

\[
\frac{dx}{dt} = \dot{x}(t) = g(x(t), u(t)), \quad t \geq 0
\]

\[
x(0) = x_0 \text{ given}
\]

and, in some cases, the terminal constraint

\[
\lim_{t \to \infty} e^{-\rho t} x(t) \geq 0
\]

By applying the principle of the dynamic programming the first order conditions for this problem are represented by the HJB equation

\[
\rho V(x) = \max_u \left\{ f(u, x) + V'(x) g(u, x) \right\}.
\]

Again, if an optimal control exists it is determined from the policy function \(u^* = h(x)\) and the HJB equation is equivalent to the functional differential equation

\[
\rho V(x) = f(h(x), x) + V'(x) g(h(x), x).
\]

Again, if we can find \(V(x)\) we can also find \(h(x)\) and can determine the optimal flow \([u^*(t), x^*(t)]_{t \in \mathbb{R}^+}\) from solving the ordinary differential equation \(\dot{x} = g(h(x), x)\) given \(x(0) = x_0\).

\[\text{We use the convention } \dot{x} = dx/dt, \text{ if } x = x(t), \text{ is a time-derivative and } V'(x) = dV/dx, \text{ if } V = V(x) \text{ is the derivative for any other argument.}\]
1.1.3 Discrete time stochastic models

The variables are random sequences \( \{u_t(\omega), x_t(\omega)\}_{t=0}^{\infty} \) which are adapted to the filtration \( \mathcal{F} = \{\mathcal{F}_t\}_{t=0}^{\infty} \) over a probability space \((\Omega, \mathcal{F}, P)\). The domain of the variables is \( \omega \in \mathbb{N} \times (\Omega, \mathcal{F}, P, \mathbb{F}) \), such that \((t, \omega) \mapsto u_t\) and \(x_t \in \mathbb{R}\) where \((t, \omega) \mapsto x_t\). Then \(u_t \in \mathbb{R}\) is a random variable.

An economic agent chooses a random sequence \( \{u_t^*, x_t^*\}_{t=0}^{\infty} \) that maximizes the sum

\[
\max_u E_0 \left[ \sum_{t=0}^{\infty} \beta^t f(u_t, x_t) \right]
\]

subject to the contingent sequence of budget constraints

\[
x_{t+1} = g(x_t, u_t, \omega_{t+1}), \quad t = 0, 1, \ldots, \infty,
\]

\[
x_0 \quad \text{given}
\]

where \(0 < \beta < 1\).

By applying the principle of the dynamic programming the first order conditions of this problem are given by the HJB equation

\[
V(x_t) = \max_u \left\{ f(u_t, x_t) + \beta E_t[V(g(u_t, x_t, \omega_{t+1}))] \right\}
\]

where \(E_t[V(g(u_t, x_t, \omega_{t+1}))] = E[V(g(u_t, x_t, \omega_{t+1}))|\mathcal{F}_t]\). If it exists, the optimal control can take the form \(u_t^* = f(E_t[v(x_{t+1})])\).

1.1.4 Continuous time stochastic models

The most common problem used in economics and finance is the following: in the space of the flows \([(u(\omega, t), x(\omega, t)) : \omega = \omega(t) \in (\Omega, \mathcal{F}, P, \mathcal{F}(t)), \quad t \in \mathbb{R}_+]\) choose a flow \([u^*(t)]\) that maximizes the functional

\[
V[u] = E_0 \left[ \int_0^{\infty} f(u(t), x(t)) e^{-\rho t} dt \right]
\]
where $u(t) = u(\omega(t), t)$ and $x(t) = x(\omega(t), t)$ are Itô processes and $\rho > 0$, such that the instantaneous budget constraint is represented by a stochastic differential equation

$$
\frac{dx}{dt} = g(x(t), u(t), t)dt + \sigma(x(t), u(t))dB(t), \ t \in \mathbb{R}_+
$$

$$
x(0) = x_0 \text{ given}
$$

where $[dB(t) : t \in \mathbb{R}_+]$ is a Wiener process.

By applying the stochastic version of the principle of DP the HJB equation is a second order functional equation

$$
\rho V(x) = \max_u \left\{ f(u, x) + g(u, x)V'(x) + \frac{1}{2}\sigma(u, x)^2 V''(x) \right\}.
$$

### 1.2 References

First contribution: Bellman (1957)


Continuous time: Fleming and Rishel (1975), Kamien and Schwartz (1991), Bertsekas (2005a), Bertsekas (2005b)
Part I

Deterministic Dynamic Programming
Chapter 2

Discrete Time

2.1 The Optimal control problem

2.1.1 Elements of the problem

- assume that time evolves in a discrete way, meaning that $t \in \{0, 1, 2, \ldots\}$, that is $t \in \mathbb{N}_0$;

- the economy is described by two variables that evolve along time: a state variable $x_t$ and a control variable, $u_t$;

- we know the initial value of the state variable, $x_0$, and the law of evolution of the state, which is a function of the control variable ($u(.)$): $x_{t+1} = g_t(x_t, u_t)$;

- in some cases, we can impose terminal conditions on the horizon, $T$, or on the state variable $v(x_T) = v_T$ with $v_T$ known;

- we assume that there are $m$ control variables, and that they belong to the set $u_t \in \mathcal{U} \subset \mathbb{R}^m$ for any $t \in \mathbb{N}_0$. 

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• then, for any sequence of controls, \( u \)
\[
u \equiv \{u_0, u_1, \ldots : u_t \in U\}
\]
the economy can follow a large number of feasible paths,
\[
x \equiv \{x_0, x_1, \ldots \}, \text{ with } x_{t+1} = g_t(x_t, u_t), \ u_t \in U
\]

• however, if we have a criteria that allows for the evaluation of all feasible paths
\[
U(x_0, x_1, \ldots, u_0, u_1, \ldots)
\]

• and if there is at least an optimal control \( u^* = \{u^*_0, u^*_1, \ldots \} \) which maximizes \( U \),

• then there is at least an optimal path for the state variable
\[
x^* \equiv \{x_0, x^*_1, \ldots \}
\]

2.1.2 The simple cake-eating problem

AS an illustration, assume that there is a cake (or a resource) with initial size \( \phi_0 > 0 \) and that we want it to have size \( \phi_T < \phi_0 \) in period \( T \). Let us denote the size of the cake at the beginning of period time \( t \) by \( W_t \). Then we know the initial and the terminal values of \( W_t, W_0 = \phi_0 \), and \( W_T = \phi_T \). The cake (resource) is consumed (depleted) in every period. Denote by \( C_t \) the consumption of the cake in period \( t \). Then
\[
\begin{align*}
W_0 &= \phi_0 \\
W_1 &= W_0 - C_0 = \phi_0 - C_0 \\
& \quad \vdots \\
W_{T-1} &= W_{T-2} - C_{T-2} = \phi_0 - \sum_{t=0}^{T-2} C_t \\
W_T &= \phi_T.
\end{align*}
\]
Therefore, in the interval between 0 and $T$ the sequence of \{$C_0, \ldots, C_{T-1}\}$ or, compactly \{$C_t\}_{t=0}^{T-1}$ is consumed. Consuming the cake, will imply that its size along time is given by the sequence \{$W_0, \ldots, W_t, \ldots, W_T\}$. We can derive the sequence of \{$W_t\}_{t=0}^T$ from the difference equation

\[ W_{t+1} = W_t - C_t, \quad t = 0, \ldots, T - 1 \]

(2.1)

together with the initial and terminal values. It has the solution

\[ W_t = W_0 - \sum_{s=0}^{t-1} C_s, \quad t = 1, \ldots, T \]

How should we eat the cake (deplete the resource) ?

Two observations: First, the solution path for $W$ depends on the path for $C$. Second, the problem seems to be mispecified because there is an infinite number of paths for $C$, which verify the constraints of the problem

\[ \sum_{t=0}^{T-1} C_s = \phi_0 - \phi_T. \]

For instance all the following are feasible solutions:

- consume everything at once \{$C_t\}_{t=0}^{T-1} = \{\phi_0 - \phi_T, 0, \ldots, 0\}$ then \n
  \[ \{W_t\}_{t=0}^T = \{\phi_0, \phi_T, \phi_T, \ldots, \phi_T\} \]

- consume everything at the end \{$C_t\}_{t=0}^{T-1} = \{0, 0, \ldots, \phi_0 - \phi_T\}$, then \n
  \[ \{W_t\}_{t=0}^T = \{\phi_0, \phi_0, \ldots, \phi_0, \phi_T\} \]

- stationary consumption strategy, $C_t = \frac{\phi_0 - \phi_T}{T}$, then $W_t = \frac{1}{T} ((T - t)\phi_0 + t\phi_T)$ then \n
  \[ \{W_t\}_{t=0}^T = \{\phi_0, \frac{1}{T} ((T - 1)\phi_0 + \phi_T), \ldots, \frac{1}{T} (\phi_0 + (T - 1)\phi_T), \phi_T\} \]
It is apparent that the specification of the cake-eating problem is incomplete. In order to choose a path for the cake (resource) we need an optimality criteria.

A simple criterium could be to maximize the sum of the consumption

$$\max_{\{C\}} \sum_{t=0}^{T-1} C_t$$

But it is simple to see that as $\sum_{t=0}^{T-1} C_t = \phi_0 - \phi_T$ then all the previous paths are solutions. That is, there is an infinite number of solutions.

Next we will see that criteria like

$$\max_{\{C\}} \sum_{t=0}^{T-1} \beta^t f(C_t)$$

for $0 < \beta < 1$ and $f(C)$ increasing and concave, will produce an unique optimal path, among the feasible paths.

If we substitute equation (2.1) at the objective functional, we get a calculus of variations problem

$$\max_{\{W\}} \sum_{t=0}^{T-1} \beta^t f(W_t - W_{t+1}).$$
### 2.1.3 Calculus of variations problems

**The simplest problem**

The simplest calculus of variations problem consists in finding the maximum or the minimum of a functional over $x \equiv \{ x_t \}_{t=0}^T$, given initial and a terminal values $x_0$ and $x_T$.

Assume that $F(x', x)$ is continuous and differentiable in $(x', x)$. The simplest problem of the calculus of variations is to maximize the value functional

$$\max_x \sum_{t=0}^{T-1} F(x_{t+1}, x_t, t)$$

(2.2)

where function $F(.)$ is called the objective function

subject to $x_0 = \phi_0$ and $x_T = \phi_T$

(2.3)

where $\phi_0$ and $\phi_T$ are given.

We denote the solution of the calculus of variations by $\{ x^*_t \}_{t=0}^T$.

The value functional is defined as

$$V(\{x\}) = \sum_{t=0}^{T-1} F(x_{t+1}, x_t, t)$$

and the optimal value functional is

$$V(\{x^*\}) = \sum_{t=0}^{T-1} F(x^*_{t+1}, x^*_t, t) = \max_{\{x\}} \sum_{t=0}^{T-1} F(x_{t+1}, x_t, t).$$

**Proposition 1. (Necessary condition for optimality)**

Let $\{x^*_t\}_{t=0}^T$ be a solution for the problem defined by equations (2.2) and (2.3). Then it verifies the Euler-Lagrange equation

$$\frac{\partial F(x^*_t, x^*_{t-1}, t-1)}{\partial x_t} + \frac{\partial F(x^*_{t+1}, x^*_t, t)}{\partial x_t} = 0, \quad t = 1, 2, \ldots, T - 1$$

(2.4)

and the initial and the terminal conditions

$$\begin{cases}
  x^*_0 = \phi_0, & t = 0 \\
  x^*_T = \phi_T, & t = T.
\end{cases}$$
Proof. Assume that we know the optimal solution \( \{x_t^*\}_{t=0}^T \). Therefore, we also know the optimal value function \( V(x^*) = \sum_{t=0}^{T-1} F(x_{t+1}^*, x_t^*, t) \). Consider an alternative candidate for solution \( \{x_t\}_{t=0}^{T-1} \) such that \( x_t = x_t^* + \varepsilon_t \), where \( \varepsilon_t \neq 0 \) for \( t = 1, \ldots, T - 1 \) but \( \varepsilon_0 = \varepsilon_T = 0 \). That is, the alternative candidate solution departs and reaches the same values of the optimal solution, but through a different path. In this case the value function is \( V(x) = \sum_{t=0}^{T-1} F(x_{t+1}, x_t, t) \). Observe that

\[
V(\{x\}) = F(x_1, x_0, 0) + F(x_2, x_1, 1) + \ldots + F(x_t, x_{t-1}, t-1) + F(x_{t+1}, x_t, t) + \ldots + F(x_T, x_{T-1}, T-1)
\]

If \( F(.) \) is differentiable, the variation in the value function is \(^1\)

\[
V(x) - V(x^*) = \left( \frac{\partial F(x_0^*, x_1^*, 0)}{\partial x_1} + \frac{\partial F(x_2^*, x_1^*, 1)}{\partial x_1} \right) \varepsilon_1 + \ldots + \left( \frac{\partial F(x_{T-1}^*, x_{T-2}^*, T-2)}{\partial x_{T-1}} + \frac{\partial F(x_T^*, x_{T-1}^*, T-1)}{\partial x_{T-1}} \right) \varepsilon_{T-1}
\]

that is

\[
V(x) - V(x^*) = \sum_{t=1}^{T-1} \left( \frac{\partial F(x_t^*, x_{t-1}^*, t-1)}{\partial x_t} + \frac{\partial F(x_{t+1}^*, x_t^*, t+1)}{\partial x_t} \right) \varepsilon_t.
\]

If \( \{x_t\}_{t=0}^{T-1} \) is an optimal solution then \( V(\{x\}) = V(\{x^*\}) \), which holds if (2.4) is verified. □

Observations

- if we have a minimum problem we have just to consider the symmetric of the value function

\[
\min_y \sum_{t=0}^{T-1} F(y_{t+1}, y_t, t) = \max_y \sum_{t=0}^{T-1} -F(y_{t+1}, y_t, t)
\]

- If \( F(x, y) \) is concave then the necessary conditions are also sufficient.

\(^1\) Recall that \( \varepsilon_0 = \varepsilon_T = 0 \).
Example: The cake eating problem for a logarithmic utility  The problem is to find the optimal paths $C^* = \{C_t^*\}_{t=0}^{T-1}$ and $W^* = \{W_t^*\}_{t=0}^T$ that solve the problem

$$\max_C \sum_{t=0}^{T-1} \beta^t \ln(C_t), \text{ subject to } W_{t+1} = W_t - C_t, \ W_0 = \phi, \ W_T = 0. \quad (2.5)$$

This problem can be transformed into the calculus of variations problem, because $C_t = W_t - W_{t+1}$,

$$\max_W \sum_{t=0}^{T-1} \beta^t \ln(W_t - W_{t+1}), \text{ subject to } W_0 = \phi, \ W_T = 0.$$  

The Euler equation is

$$-\beta^{t-1} \frac{1}{W_{t-1} - W_t} + \beta^t \frac{1}{W_t - W_{t+1}} = 0, \ t = 1, 2, \ldots, T - 1$$

or, equivalently

$$W_{t+2}^* = (1 + \beta)W_{t+1}^* - \beta W_t^*, \ t = 0, \ldots T - 2.$$  

has the general solution

$$W_t^* = \frac{1}{1 - \beta} (-\beta k_1 + k_2 + (k_1 - k_2)\beta^t), \ t = 0, 1, \ldots, T$$

which depends on two arbitrary constants, $k_1$ and $k_2$. We can evaluate them by using the initial and terminal conditions

$$\begin{cases}  
W_0^* = \frac{1}{1 - \beta} (-\beta k_1 + k_2 + (k_1 - k_2)) = \phi \\
W_T^* = -\beta k_1 + k_2 + (k_1 - k_2)\beta^T = 0,
\end{cases}$$

then

$$k_1 = \phi, \ k_2 = \frac{\beta - \beta^T}{1 - \beta^T}\phi.$$  

Therefore, the solution for the cake-eating problem $C^*, W^*$ is generated by

$$W_t^* = \left(\frac{\beta^t - \beta^T}{1 - \beta^T}\right) \phi, \ t = 0, 1, \ldots T  \quad (2.6)$$

and, as $C_t^* = W_t^* - W_{t+1}^*$

$$C_t^* = \left(\frac{1 - \beta}{1 - \beta^T}\right) \beta^t \phi, \ t = 0, 1, \ldots T - 1.  \quad (2.7)$$
The free terminal state problem

Now let us consider the problem

$$\max_x \sum_{t=0}^{T-1} F(x_{t+1}, x_t, t)$$

subject to $x_0 = \phi_0$ and $x_T$ free

(2.8)

where $\phi_0$ is given.

**Proposition 2.** (Necessary condition for optimality for the free end point problem)

Let $\{x_t^*\}_{t=0}^T$ be a solution for the problem defined by equations (2.2) and (2.3). Then it verifies the Euler-Lagrange equation

$$\frac{\partial F(x_t^*, x_{t-1}^*, t-1)}{\partial x_t} + \frac{\partial F(x_{t+1}^*, x_t^*, t)}{\partial x_t} = 0, \quad t = 1, 2, \ldots, T - 1$$

(2.9)

and the initial and the transversality condition

\[
\begin{cases}
  x_0^* = \phi_0, & t = 0 \\
  \frac{\partial F(x_T^*, x_{T-1}^*, T-1)}{\partial x_T} = 0, & t = T.
\end{cases}
\]

**Proof.** Again we assume that we know $\{x_t^*\}_{t=0}^T$ and $V(\{x^*\})$, and we use the same method as in the proof for the simplest problem, with the difference that the perturbation for the final period is $\epsilon_T \neq 0$. Now

$$V(x) - V(x^*) = \sum_{t=1}^{T-1} \left( \frac{\partial F(x_t^*, x_{t-1}^*, t-1)}{\partial x_t} x_t + \frac{\partial F(x_{t+1}^*, x_t^*, t+1)}{\partial x_t} x_t \right) \epsilon_t + \frac{\partial F(x_T^*, x_{T-1}^*, T-1)}{\partial x_T} \epsilon_T$$

Then $V(x) - V(x^*) = 0$ if and only if the Euler and the transversality conditions are verified. $$\square$$
Condition (2) is called the **transversality condition**. Its meaning is the following: if the terminal state of the system is free, it would be optimal if there is no gain in changing the solution trajectory as regards the horizon of the program. If \( \frac{\partial F(x^*_T, x^*_{T-1}, T-1)}{\partial x_T} > 0 \) then we could improve the solution by increasing \( x^*_T \) (remember that the utility functional is additive along time) and if \( \frac{\partial F(x^*_T, x^*_{T-1}, T-1)}{\partial x_T} < 0 \) we have a non-optimal terminal state by excess.

However, in free endpoint problems we need an additional terminal condition in order to have a meaningful solution. To convince oneself, consider the following problem.

**Cake eating problem with free terminal size.** Consider the previous example but assume that \( T \) is known but \( W_T \) is free. The first order conditions from proposition (2.11) are

\[
\begin{align*}
W_{t+1} &= (1 + \beta)W_t - \beta W_{t-1} \\
W_0 &= \phi \\
\beta^{T-1} W_T - W_{T-1} &= 0.
\end{align*}
\]

If we substitute the solution of the Euler-Lagrange equation we get

\[
\frac{\beta^{T-1}}{W_T - W_{T-1}} = \frac{\beta^{T-1}}{\beta^T - \beta^{T-1}} \frac{1 - \beta}{k_1 - k_2} = \frac{1}{k_1 - k_2}
\]

which can only be zero if \( k_2 - k_1 = \infty \). If we look at the transversality condition, the last condition only holds if \( W_T - W_{T-1} = \infty \), which does not make sense. \( \square \)

One way to solve this, and which is very important in applications to economics is to introduce a terminal constraint.

**Free terminal state problem with a terminal constraint** Consider the problem

\[
\begin{align*}
\max_x \sum_{t=0}^{T-1} F(x_{t+1}, x_t, t) \\
\text{subject to } x_0 = \phi_0 \text{ and } x_T \geq \phi_T
\end{align*}
\]

where \( \phi_0 \) and \( \phi_T \) are given.
Proposition 3. (Necessary condition for optimality for the free end point problem with terminal constraints)

Let \( \{x_t^*\}_{t=0}^T \) be a solution for the problem defined by equations (2.2) and (2.10). Then it verifies the Euler-Lagrange equation

\[
\frac{\partial F(x_t^*, x_{t-1}^*, t-1)}{\partial x_t} + \frac{\partial F(x_{t+1}^*, x_t^*, t)}{\partial x_t} = 0, \quad t = 1, 2, \ldots, T - 1 \tag{2.11}
\]

and the initial and the transversality condition

\[
\begin{cases}
    x_0^* = \phi_0, & t = 0 \\
    \frac{\partial F(x_t^*, x_{t-1}^*, T-1)}{\partial x_T}(\phi_T - x_T) = 0, & t = T.
\end{cases}
\]

Proof. Now we write \( V(x) \) as a Lagrangean

\[
V(x) = \sum_{t=0}^{T-1} F(x_{t+1}, x_t, t) + \lambda(\phi_T - x_T)
\]

where \( \lambda \) is a lagrange multiplier. Using again the variation method with \( \epsilon_0 = 0 \) and \( \epsilon_T \neq 0 \) we have

\[
V(x) - V(x^*) = \sum_{t=1}^{T-1} \left( \frac{\partial F(x_t^*, x_{t-1}^*, t-1)}{\partial x_t} + \frac{\partial F(x_{t+1}^*, x_t^*, t+1)}{\partial x_t} \right) \epsilon_t + \\
\frac{\partial F(x_T^*, x_{T-1}^*, T-1)}{\partial x_T} \epsilon_T - \lambda(x_T - x_T^*)
\]

From the Kuhn-Tucker conditions, we have the conditions, regarding the terminal state,

\[
\frac{\partial F_{T-1}}{\partial x_T} - \lambda = 0, \quad \lambda(\phi_T - x_T^*) = 0.
\]

\qed
The cake eating problem again  Now, if we introduce the terminal condition \( W_T \geq 0 \), the first order conditions are

\[
W_{t+1} = (1 + \beta)W_t - \beta W_{t+1} \\
W_0 = \phi \\
\frac{\beta^{T-1}W_T}{W_T - W_{T-1}} = 0.
\]

If \( T \) is finite, the last condition only holds if \( W_T = 0 \), which means that it is optimal to eat all the cake in finite time. The solution is, thus formally, but not conceptually, the same as in the fixed endpoint case.

Infinite horizon problem

The most common problems in macroeconomics is the discounted infinite horizon problem:

\[
\max_x \sum_{t=0}^{\infty} \beta^{t-1}F(x_{t+1}, x_t) \tag{2.12}
\]

where, \( 0 < \beta < 1 \), \( x_0 = \phi_0 \) where \( \phi_0 \) is given.

**Proposition 4. (Necessary condition for optimality for the infinite horizon problem)**

Let \( \{x_t^{*}\}_{t=0}^{T} \) be a solution for the problem defined by equations (2.12) and (2.3). Then it verifies the Euler-Lagrange equation

\[
\frac{\partial F(x_t^{*}, x_{t-1}^{*})}{\partial x_t} + \beta \frac{\partial F(x_{t+1}, x_t)}{\partial x_t} = 0, \ t = 0, 1, \ldots
\]

and

\[
\begin{aligned}
x_0^{*} &= x_0, \\
\lim_{t \to \infty} \beta^{t-1} \frac{\partial F(x_t^{*}, x_{t-1}^{*})}{\partial x_t} &= 0,
\end{aligned}
\]
Infinite horizon with terminal conditions If we assume that $\lim_{t \to \infty} x_t = 0$ then the transversality condition becomes

$$\lim_{t \to \infty} \beta^t \frac{\partial F(x_t^*, x_{t-1}^*)}{\partial x_t} x_t^* = 0.$$ 

The cake eating problem The solution of the Euler-Lagrange equation was already derived as

$$W_t^* = \frac{1}{1 - \beta} \left( -\beta k_1 + k_2 + (k_1 - k_2)\beta^t \right), \quad t = 0, 1, \ldots, \infty$$

If we substitute in the transversality condition for the infinite horizon problem without terminal conditions, we get

$$\lim_{t \to \infty} \beta^t \frac{\partial \ln(W_{t-1} - W_t)}{\partial W_t} W_t = \lim_{t \to \infty} \beta^t (W_t - W_{t-1})^{-1} = \lim_{t \to \infty} \beta^{t-1} \frac{1 - \beta}{\beta^t - \beta^{t-1} k_1 - k_2} = \frac{1}{k_2 - k_1}$$

which again is only zero if $k_2 - k_1 = \infty$. If we consider the infinite horizon problem with a terminal constraint $\lim_{t \to \infty} W_t \geq 0$ and substitute in the transversality condition for the infinite horizon problem without terminal conditions, we get

$$\lim_{t \to \infty} \beta^{t-1} \frac{\partial \ln(W_{t-1} - W_t)}{\partial W_t} W_t = \lim_{t \to \infty} \frac{W_t}{k_1 - k_2} = \frac{-\beta k_1 + k_2}{(1 - \beta)(k_2 - k_1)}$$

because $\lim_{t \to \infty} \beta^t = 0$ because $0 < \beta < 1$. The transversality condition holds if and only if $k_2 = \beta k_1$. If we substitute in the solution for $W_t$, we get

$$W_t^* = \frac{k_1 (1 - \beta)}{1 - \beta} \beta^t = k_1 \beta^t, \quad t = 0, 1, \ldots, \infty$$

and, as the solution should verify the initial condition $W_0 = \phi_0$ we finally get the solution for the infinite horizon problem

$$W_t^* = \phi_0 \beta^t, \quad t = 0, 1, \ldots, \infty$$
2.1.4 Optimal control problems

The most common dynamic optimization problems in economics and finance have the following common assumptions

- timing: the state variable $x_t$ is usually a stock and is measured at the beginning of period $t$ and the control $u_t$ is usually a flow and is measured in the end of period $t$;

- horizon: can be finite or is infinite ($T = \infty$). The second case is more common;

- objective functional:
  - there is an intertemporal utility function is additively separable, stationary, and involves time-discounting (impatience):
    \[
    \sum_{t=0}^{T-1} \beta^t f(u_t, x_t),
    \]
    - $0 < \beta < 1$, models impatience as $\beta^0 = 1$ and $\lim_{t \to \infty} \beta^t = 0$;
    - $f(\cdot)$ is well behaved: it is continuous, continuously differentiable and concave in $(u, x)$;

- the economy is described by an autonomous difference equation
  \[
  x_{t+1} = g(x_t, u_t)
  \]
  where $g(\cdot)$ is autonomous, continuous, differentiable, concave. Then the DE verifies the conditions for existence and uniqueness of solutions;

- the non-Ponzi game condition holds:
  \[
  \lim_{t \to \infty} \varphi^t x_t \geq 0
  \]
  holds, for a discount factor $0 < \varphi < 1$;

- there may be some side conditions, e.g., $x_t \geq 0$, $u_t \geq 0$, which may produce corner solutions. We will deal only with the case in which the solutions are interior (or the domain of the variables is open).
These assumptions are formalized as optimal control problems:

Definition 1. The simplest optimal control problem (OCP): Find \( \{u_t^*, x_t\}_{t=0}^T \) which solves

\[
\max_{\{u_t\}_{t=0}^T} \sum_{t=0}^{T} \beta^t f(u_t, x_t)
\]

such that \( u_t \in U \) and

\[ x_{t+1} = g(x_t, u_t) \]

for \( x_0, x_T \) given and \( T \) free.

Definition 2. The free terminal state optimal control problem (OCP): Find \( \{u_t^*, x_t\}_{t=0}^{T-1} \), which solves

\[
\max_{\{u_t\}_{t=0}^{T-1}} \sum_{t=0}^{T} \beta^t f(u_t, x_t)
\]

such that \( u_t \in U \) and

\[ x_{t+1} = g(x_t, u_t) \]

for \( x_0, T \) given and \( x_T \) free.

If \( T = \infty \) we have the infinite horizon discounted optimal control problem

Methods for solving the OCP in the sense of obtaining necessary conditions or necessary and sufficient conditions:

- method of Lagrange (for the case \( T \) finite)
- Pontryagin’s maximum principle (Pontryagin et al. (1962))
- Dynamic programming principle (Bellman (1957)).

Necessary and sufficient optimality conditions Intuitive meaning:
• necessary conditions: assuming that we know the optimal solution, \( \{u_t^*, x_t^*\} \) which optimality conditions should the variables of the problem verify? (This means that they hold for every extremum feasible solutions);

• sufficient conditions: if the functions defining the problem, \( f(\cdot) \) and \( g(\cdot) \), verify some conditions, then feasible paths verifying some optimality conditions are solutions of the problem.

In general, if the behavioral functions \( f(\cdot) \) and \( g(\cdot) \) are well behaved (continuous, continuously differentiable and concave) then necessary conditions are also sufficient.
2.1.5 Dynamic programming

The Principle of dynamic programming (Bellman, 1957, p. 83):

\begin{quote}
\textit{an optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision.}
\end{quote}

We next follow an heuristic approach for deriving necessary conditions for problem 1, following the principle of DP.

Finite horizon problem  Assume that we know a solution optimal control \( \{u^*, x^*_t\}_{t=0}^T \).

Which properties should the optimal solution have?

Definition 3. Definition: value function for time \( \tau \)

\[
V_\tau(x_\tau) = \sum_{t=\tau}^{T-1} \beta^{t-\tau} f(u^*_t, x^*_t)
\]

Proposition 5. Given an optimal solution to the optimal control problem, solution optimal control \( \{u^*, x^*_t\}_{t=0}^T \), then it verifies Hamilton-Jacobi-Equation

\[
V_t(x_t) = \max_{u_t} \left\{ f(x_t, u_t) + \beta V_{t+1}(x_{t+1}), \ t = 0, \ldots, T - 1 \right\}
\] (2.13)
Proof. If we know a solution for problem 1, then at time $\tau = 0$, we have

$$V_0(x_0) = \sum_{t=0}^{T-1} \beta^t f(u^*_t, x^*_t) =$$

$$= \max_{\{u_t\}_{t=0}^{T-1}} \sum_{t=0}^{T-1} \beta^t f(u_t, x_t) =$$

$$= \max_{\{u_t\}_{t=0}^{T-1}} \left( f(x_0, u_0) + \beta f(x_1, u_1) + \beta^2 f(x_2, u_2) + \ldots \right) =$$

$$= \max_{\{u_t\}_{t=0}^{T-1}} \left( f(x_0, u_0) + \beta \sum_{t=1}^{T-1} \beta^{t-1} f(x_t, u_t) \right) =$$

$$= \max_{u_0} \left( f(x_0, u_0) + \beta \max_{\{u_t\}_{t=1}^{T-1}} \sum_{t=1}^{T-1} \beta^{t-1} f(x_t, u_t) \right)$$

by the principle of dynamic programming. Then

$$V_0(x_0) = \max_{u_0} \{ f(x_0, u_0) + \beta V_1(x_1) \}$$

We can apply the same idea for the value function for any time $0 \leq t \leq T$ to get equation (2.13), which holds for feasible solutions, i.e., verifying $x_{t+1} = g(x_t, u_t)$ and given $x_0$. \qed

Intuition: we transform the maximization of a functional into a recursive two-period problem. We solve the control problem by solving the HJB equation. To do this we have to find the sequence $\{V_0, \ldots, V_T\}$, through the recursion

$$V_t(x) = \max_u \{ f(x, u) + \beta V_{t+1}(g(x, u)) \} \quad (2.14)$$

Infinite horizon problem

Definition 4. The infinite horizon optimal control problem (OCP): Find $\{u^*_t, x_t\}_{t=0}^{\infty}$, which solves

$$\max_{\{u_t\}_{t=0}^{\infty}} \sum_{t=0}^{T} \beta^t f(u_t, x_t)$$
such that
\[ x_{t+1} = g(x_t, u_t) \]
for \( x_0 \) given.

For the infinite horizon discounted optimal control problem, the limit function \( V = \lim_{j \to \infty} V_j \) is independent of \( j \) so the Hamilton Jacobi Bellman equation becomes
\[
V(x) = \max_u \{ f(x, u) + \beta V[g(x, u)] \} = \max_u H(x, u)
\]

**Properties of the value function:** it is usually difficult to get the properties of \( V(.) \). In general continuity is assured but not differentiability (this is a subject for advanced courses on DP, see Stokey and Lucas (1989)).

If some regularity conditions hold, we may determine the optimal control through the **optimality condition**
\[
\frac{\partial H(x, u)}{\partial u} = 0
\]
if \( H(.) \) is \( C^2 \) then we get the **policy function**
\[
u^* = h(x)
\]
which gives an optimal rule for changing the optimal control, given the state of the economy. If we can determine (or prove that there exists such a relationship) then we say that our problem is **recursive**.

In this case the HJB equation becomes a non-linear functional equation
\[
V(x) = f(x, h(x)) + \beta V[g(x, h(x))].
\]

**Solving the HJB:** means finding the value function \( V(x) \). Methods: analytical (in some cases exact) and mostly numerical (value function iteration).
2.2 Applications

2.2.1 The cake eating problem

We consider the infinite horizon problem

$$\max_{\{C_t\}_{t=0}^\infty} \sum_{t=0}^{T} \beta^t \ln(C_t)$$

subject to

$$W_{t+1} = W_t - C_t$$

for $W_0 = \phi$ given.

The Hamilton-Jacobi-Bellman equation is

$$V(W) = \max_C \{\ln(C) + \beta V(W - C)\}.$$ 

We will solve the problem if we can find the unknown function $V(W)$.

The optimality condition is

$$\frac{\partial \{\ln(C) + \beta V(W - C)\}}{\partial C} = \frac{1}{C} - \beta V'(W - C) = 0$$

which we express as

$$C = \frac{1}{\beta V'(W - C)}.$$ 

Possibly we can solve uniquely for $C$ to get $C^* = C(W)$, and $W_1(W) = W - C(W)$. Then the HJB becomes a functional equation

$$V(W) = \ln(C^*(W)) + \beta V(W_1(W)).$$

We will solve it by using the **method of the undetermined coefficients**. Assume that the solution has the form

$$V(W) = a + b \ln(W)$$
where the coefficients \(a\) and \(b\) are unknown. It is indeed a solution, if, upon substituting it in the HJB equation, we could determine \(a\) and \(b\) as functions of known parameters.

First, we find that

\[
C = \frac{1}{1 + b\beta} W
\]

\[
W_1 = \frac{b\beta}{1 + b\beta} W
\]

Substituting in the HJB equation, we get

\[
a + b \ln(W) = \ln(W) - \ln(1 + b\beta) + \beta \left( a + b \ln\left( \frac{b\beta}{1 + b\beta} \right) + b \ln(W) \right),
\]

which is equivalent to

\[
(b(1 - \beta) - 1) \ln(W) = a(\beta - 1) - \ln(1 + b\beta) + \beta b \ln\left( \frac{b\beta}{1 + b\beta} \right).
\]

We determine \(b\) and \(a\) by setting both terms to zero

\[
b = \frac{1}{1 - \beta}
\]

\[
a = \ln(1 - \beta) + \frac{\beta}{1 - \beta} \ln(\beta)
\]

Then the value function is

\[
V(W) = \frac{1}{1 - \beta} \ln(\chi W), \text{ where } \chi \equiv (\beta^\beta (1 - \beta)^{1-\beta})^{1/(1-\beta)}.
\]

and

\[
C = (1 - \beta)W.
\]

Therefore, we determine the optimal paths by substituting \(C\) into the restriction of the problem,

\[
W_{t+1} = W_t - (1 - \beta)W_t = \beta W_t
\]

and solving it

\[
W_t = \phi \beta^t, \ t = 0, \ldots, \infty
\]
and

\[ C_t = \phi(1 - \beta)\beta^t, \quad t = 1, \ldots, \infty. \]

### 2.2.2 The consumption-portfolio choice problem

**Assumptions:**

- there are \( T > 1 \) periods;
- consumers are homogeneous and have an additive intertemporal utility functional;
- the instantaneous utility function is continuous, differentiable, increasing, concave and is homogenous of degree \( n \);
- consumers have a stream of endowments, \( y \equiv \{y_t\}_{t=0}^{T} \), known with certainty;
- institutional setting: there are spot markets for the good and for a financial asset. The financial asset is an entitlement to receive the dividend \( D_t \) at the end of every period \( t \). The spot prices are \( P_t \) and \( S_t \) for the good and for the financial asset, respectively.
- Market timing, we assume that the good market opens in the beginning and that the asset market opens at the end of every period.

**The consumer’s problem**

- choose a sequence of consumption \( \{c_t\}_{t=0}^{T} \) and of portfolios \( \{\theta_t\}_{t=1}^{T} \), which is, in this simple case, the quantity of the asset bought at the beginning of time \( t \), in order to find

\[
\max_{\{c_t, \theta_{t+1}\}^T_{t=0}} \sum_{t=0}^{T-1} \beta^t u(c_t)
\]
subject to the sequence of budget constraints:

\[
\begin{align*}
A_0 + y_0 &= c_0 + \theta_1 S_0 \\
\theta_1 (S_1 + D_1) + y_1 &= c_1 + \theta_2 S_1 \\
&\quad \cdots \\
\theta_t (S_t + D_t) + y_t &= c_t + \theta_{t+1} S_t \\
&\quad \cdots \\
\theta_T (S_T + D_T) + y_T &= c_T
\end{align*}
\]

where \( A_t \) is the stock of financial wealth (in real terms) at the beginning of period \( t \).

If we denote

\[ A_{t+1} = \theta_{t+1} S_t \]

then the generic period budget constraint is

\[ A_{t+1} = y_t - c_t + R_t A_t, \quad t = 0, \ldots, T \] (2.15)

where the asset return is

\[ R_{t+1} = 1 + r_{t+1} = \frac{S_{t+1} + D_{t+1}}{S_t}. \]

Then the HJB equation is

\[ V(A_t) = \max_{c_t} \left\{ u(c_t) + \beta V(A_{t+1}) \right\} \] (2.16)

We can write the equation as

\[ V(A) = \max_c \left\{ u(c) + \beta V(\tilde{A}) \right\} \] (2.17)

where \( \tilde{A} = y - c + RA \) then \( V(\tilde{A}) = V(y - c + RA) \).
Deriving an intertemporal arbitrage condition

The optimality condition is:

$$u'(c^*) = \beta V'(\tilde{A})$$

if we could find the optimal policy function $c^* = h(A)$ and substitute it in the HJB equation we would get

$$V(A) = u(h(A)) + \beta V(\tilde{A}^*), \quad \tilde{A}^* = y - h(A) + RA.$$  

Using the Benveniste and Scheinkman (1979) envelope theorem, we differentiate for $A$ to get

$$V'(A) = u'(c^*) \frac{\partial h}{\partial A} + \beta V'(\tilde{A}^*) \frac{\partial \tilde{A}^*}{\partial A} =$$

$$= u'(c^*) \frac{\partial h}{\partial A} + \beta V'(\tilde{A}^*) \left( R - \frac{\partial h}{\partial A} \right) =$$

$$= \beta RV'(\tilde{A}^*) =$$

$$= Ru'(c^*)$$

from the optimality condition. If we shift both members of the last equation we get

$$V'(\tilde{A}^*) = Ru'(\tilde{c}^*),$$

and, then

$$Ru'(\tilde{c}^*) = \beta^{-1} u'(c^*).$$

Then, the optimal consumption path (we delete the * from now on) verifies the arbitrage condition

$$u'(c) = \beta Ru'(\tilde{c}).$$

In the literature the relationship is called the consumer’s intertemporal arbitrage condition

$$u'(c_t) = \beta u'(c_{t+1}) R_t$$  \hspace{1cm} (2.18)
Observe that
\[ \beta R_t = \frac{1 + r_t}{1 + \rho} \]
is the ratio between the market return and the psychological factor.

If the utility function is homogenous of degree \( \eta \), it has the properties
\[
\begin{align*}
    u(c) &= c^\eta u(1) \\
    u'(c) &= c^{\eta-1} u'(1)
\end{align*}
\]
the arbitrage condition is a linear difference equation
\[ c_{t+1} = \lambda c_t, \quad \lambda \equiv (\beta R_t)^{\frac{1}{1-\eta}} \]

**Determining the optimal value function**

In some cases, we can get an explicit solution for the HJB equation (2.17). We have to determine jointly the optimal policy function \( h(A) \) and the optimal value function \( V(A) \).

We will use the same non-constructive method to derive both functions: first, we make a conjecture on the form of \( V(\cdot) \) and then apply the method of the undetermined coefficients.

**Assumptions** Let us assume that the utility function is logarithmic: \( u(c) = \ln(c) \) and assume for simplicity that \( y = 0 \).

In this case the optimality condition becomes
\[ c^* = [\beta V'(RA - c^*)]^{-1} \]

**Conjecture:** let us assume that the value function is of the form
\[
V(A) = B_0 + B_1 \ln(A) \tag{2.19}
\]

\(^2\)Alternatively, if we denote \( H_t \) the human capital at the beginning of period \( t \), and \( H_{t+1} = y_t + R_t H_t \), then we could define total capital at the beginning of period \( t \) as \( W_t = A_t + H_t \) and the next results will be valid for the stock of total capital \( H_t \).
where $B_0$ and $B_1$ are undetermined coefficients.

From this point on we apply again the method of the undetermined coefficients: if the conjecture is right then we will get an equation without the independent variable $A$, and which would allow us to determine the coefficients, $B_0$ and $B_1$, as functions of the parameters of the HJB equation.

Then

$$V' = \frac{B_1}{A}.$$  

Applying this to the optimality condition, we get

$$c^* = h(A) = \frac{RA}{1 + \beta B_1}$$

then

$$\tilde{A}^* = RA - c^* = \left( \frac{\beta B_1}{1 + \beta B_1} \right) RA$$

which is a linear function of $A$.

Substituting in the HJB equation, we get

$$B_0 + B_1 \ln(A) = \ln \left( \frac{RA}{1 + \beta B_1} \right) + \beta \left[ B_0 + B_1 \ln \left( \frac{\beta B_1 RA}{1 + \beta B_1} \right) \right] = \ln \left( \frac{R}{1 + \beta B_1} \right) + \ln(A) + \beta \left[ B_0 + B_1 \ln \left( \frac{\beta B_1 R}{1 + \beta B_1} \right) + \ln(A) \right]$$

(2.20)

The term in $\ln(A)$ can be eliminated if $B_1 = 1 + \beta B_1$, that is if

$$B_1 = \frac{1}{1 - \beta}$$

and equation (2.20) reduces to

$$B_0 (1 - \beta) = \ln(R(1 - \beta)) + \frac{\beta}{1 - \beta} \ln(R\beta)$$
which we can solve for \( B_0 \) to get

\[
B_0 = (1 - \beta)^{-2} \ln(R\Theta), \text{ where } \Theta \equiv (1 - \beta)^{1-\beta} \beta^\beta
\]

Finally, as our conjecture proved to be right, we can substitute \( B_0 \) and \( B_1 \) in equation (2.19) the optimal value function and the optimal policy function are

\[
V(A) = (1 - \beta)^{-1} \ln \left( (R\Theta)^{(1-\beta)^{-1}} A \right)
\]

and

\[
c^* = (1 - \beta)RA
\]

then optimal consumption is linear in financial wealth.

We can also determine the optimal asset accumulation, by noting that \( c_t = c^* \) and substituting in the period budget constraint

\[
A_{t+1} = \beta R_t A_t
\]

If we assume that \( R_t = R \) then the solution for that DE is

\[
A_t = (\beta R)^t A_0, \; t = 0, \ldots, \infty
\]

and, therefore the optimal path for consumption is

\[
c_t = (1 - \beta)\beta^t R^{t+1} A_0
\]

Observe that the transversality condition holds,

\[
\lim_{t \to \infty} R^{-t} A_t = \lim_{t \to \infty} A_0 \beta^t = 0
\]

because \( 0 < \beta < 1 \).
Exercises

1. solve the HJB equation for the case in which \( y > 0 \)

2. solve the HJB equation for the case in which \( y > 0 \) and the utility function is CRRA:
   \[ u(c) = \frac{c^{1-\theta}}{1 - \theta}, \text{ for } \theta > 0; \]

3. try to solve the HJB equation for the case in which \( y = 0 \) and the utility function is
   CARA: \( u(c) = B - e^{-\beta c}/\beta, \text{ for } \beta > 0 \)

4. try to solve the HJB equation for the case in which \( y > 0 \) and the utility function is
   CARA: \( u(c) = B - e^{-\beta c}/\beta, \text{ for } \beta > 0. \)
2.2.3 The Ramsey problem

Find a sequence \( \{c_t, k_t\}_{t=0}^\infty \) which solves the following optimal control problem:

\[
\max_{\{c\}_{t=0}^\infty} \sum_{t=0}^\infty \beta^t u(c_t)
\]
subject to

\[k_{t+1} = f(k_t) - c_t + (1 - \delta)k_t\]

where \(0 \leq \delta \leq 1\) given \(k_0\). Both the utility function and the production function are neoclassical: continuous, differentiable, smooth and verify the Inada conditions. These conditions would ensure that the necessary conditions for optimality are also sufficient.

The HJB function is

\[V(k) = \max_c \{u(c) + \beta V(k_1)\}\]

where \(k_1 = f(k) - c + (1 - \delta)k\).

The optimality condition is

\[u'(c) = \beta V'(k_1)\]

If it allows us to find a policy function \(c = h(k)\) then the HJB becomes

\[V(k) = u[h(k)] + \beta V[f(k) - h(k) + (1 - \delta)k]\]

This equation has no explicit solution for generic utility and production function. Even for explicit utility and production functions the HJB has not an explicit solution. Next we present a benchmark case where we can find an explicit solution for the HJB equation.

**Benchmark case:** Let \(u(c) = \ln(c)\) and \(f(k) = Ak^\alpha\) for \(0 < \alpha < 1\), and \(\delta = 1\).

Conjecture:

\[V(k) = B_0 + B_1 \ln(k)\]
In this case the optimality condition is

\[ h(k) = \frac{Ak^\alpha}{1 + \beta B_1} \]

If we substitute in equation (2.21) we get

\[ B_0 + B_1 \ln(k) = \ln \left( \frac{Ak^\alpha}{1 + \beta B_1} \right) + \beta \left[ B_0 + B_1 \ln \left( \frac{Ak^\alpha B_1}{1 + \beta B_1} \right) \right] \] (2.22)

Again we can eliminate the term in \( \ln(k) \) by making

\[ B_1 = \frac{\alpha}{1 - \alpha \beta} \]

Thus, (2.22) changes to

\[ B_0(1 - \beta) = \ln(A(1 - \alpha \beta)) + \frac{\alpha \beta}{1 - \alpha \beta} \ln(\alpha \beta A) \]

Finally the optimal value function and the optimal policy function are

\[ V(A) = (1 - \alpha \beta)^{-1} \ln \left( (A \Theta)^{(1 - \beta)^{-1}} A \right), \quad \Theta \equiv (1 - \alpha \beta)^{1-\alpha \beta}(\alpha \beta)^{\alpha \beta} \]

and

\[ c^* = (1 - \alpha \beta)Ak^\alpha \]

Then the optimal capital accumulation is governed by the equation

\[ k_{t+1} = \alpha \beta Ak_t^\alpha \]

This equation generates a forward path starting from the known initial capital stock \( k_0 \)

\[ \{k_t^*\}_{t=0}^{\infty} = \{k_0, \alpha \beta A k_0^\alpha, (\alpha \beta A)^{\alpha + 1}k_0^{\alpha^2}, \ldots, (\alpha \beta A)^{\alpha^{t-1}+1}k_0^{\alpha^t}, \ldots\} \]

which converges to a stationary solution: \( \bar{k} = (\alpha \beta A)^{1/(1-\alpha)} \).
Chapter 3

Continuous Time

3.1 The dynamic programming principle and the HJB equation

3.1.1 Simplest problem optimal control problem

In the space of the functions \((u(t), x(t))\) for \(t_0 \leq t \leq t_1\) find functions \((u^*(t), x^*(t))\) which solve the problem:

\[
\max_{u(t)} \int_{t_0}^{t_1} f(t, x(t), u(t)) dt
\]

subject to

\[
\dot{x} \equiv \frac{dx(t)}{dt} = g(t, x(t), u(t))
\]

given \(x(t_0) = x_0\). We assume that \(t_1\) is know and that \(x(t_1)\) is free.

The value function is, for the initial instant

\[
\mathcal{V}(t_0, x_0) = \int_{t_0}^{t_1} f(t, x^*, u^*) dt
\]

and for the terminal time \(\mathcal{V}(t_1, x(t_1)) = 0\).
Lemma 1. First order necessary conditions for optimality from the Dynamic Programming principle

Let \( V \in C^2(\mathbb{T}, \mathbb{R}) \). Then the value function which is associated to the optimal path \( \{(x^*(t), u^*(t) : t_0 \leq t \leq t_1\} \) verifies the fundamental partial differential equation or the Hamilton-Jacobi-Bellman equation

\[
- V_t(t, x) = \max_u \left[ f(t, x, u) + V_x(t, x)g(t, x, u) \right].
\]

Proof. Consider the value function

\[
V(t_0, x_0) = \max_u \left( \int_{t_0}^{t_1} f(t, x, u) dt \right)
\]

\[
= \max_u \left( \int_{t_0}^{t_0+\Delta t} f(.) dt + \int_{t_0+\Delta t}^{t_1} f(.) dt \right) = (\Delta t > 0, \text{small})
\]

\[
= \max_u \left[ \int_{t_0}^{t_0+\Delta t} f(.) dt + \max_u \left( \int_{t_0+\Delta t}^{t_1} f(.) dt \right) \right] =
\]

\[
= \max_u \left[ \int_{t_0}^{t_0+\Delta t} f(.) dt + V(t_0 + \Delta t, x_0 + \Delta x) \right] =
\]

\[
t_0 \leq t \leq t_0 + \Delta t
\]

(From Dynamic prog priniciple)

\[
= \max_u \left[ f(t_0, x_0, u) \Delta t + V(t_0, x_0) + V_t(t_0, x_0) \Delta t + V_x(t_0, x_0) \Delta x + \text{h.o.t} \right]
\]

\[
t_0 \leq t \leq t_0 + \Delta t
\]

if \( u \approx \text{constant and } V \in C^2(\mathbb{T}, \mathbb{R}) \). Passing \( V(t_0, x_0) \) to the second member, dividing by \( \Delta t \) and taking the limit \( \lim_{\Delta t \to 0} \) we get, for every \( t \in [t_0, t_1] \),

\[
0 = \max_u \left[ f(t, x, u) + V_t(t, x) + V_x(t, x)\dot{x} \right].
\]
The policy function is now \( u^* = h(t, x) \). If we substitute in the HJB equation then we get a first order partial differential equation

\[
-\mathcal{V}_t(t, x) = f(t, x, h(t, x)) + \mathcal{V}_x(t, x)g(t, x, h(t, x)).
\]

Though the differentiability of \( \mathcal{V} \) is assured for the functions \( f \) and \( g \) which are common in the economics literature, we can get explicit solutions, for \( V(.) \) and for \( h(.) \), only in very rare cases. Proving that \( \mathcal{V} \) is differentiable, even in the case in which we cannot determine it explicitly is hard and requires proficiency in Functional Analysis.

### 3.1.2 Infinite horizon discounted problem

**Lemma 2.** First order necessary conditions for optimality from the Dynamic Programming principle

Let \( \mathcal{V} \in C^2(\mathbb{T}, \mathbb{R}) \). Then the value function associated to the optimal path \( \{(x^*(t), u^*(t) : t_0 \leq t < +\infty\} \) verifies the fundamental non-linear ODE called the Hamilton-Jacobi-Bellman equation

\[
\rho V(x) = \max_u [f(x, u) + V'(x)g(x, u)].
\]

**Proof.** Now, we have

\[
\mathcal{V}(t_0, x_0) = \max_u \left( \int_{t_0}^{+\infty} f(x, u)e^{-\rho t} dt \right) =
\]

\[
e^{-\rho t_0} \max_u \left( \int_{t_0}^{+\infty} f(x, u)e^{-\rho(t-t_0)} dt \right) =
\]

\[
e^{-\rho t_0} V(x_0) \tag{3.1}
\]

where \( V(.) \) is independent from \( t_0 \) and only depends on \( x_0 \). We can do

\[
V(x_0) = \max_u \left( \int_{0}^{+\infty} f(x, u)e^{-\rho t} dt \right).
\]
If we let, for every \((t, x)\) \(V(t, x) = e^{-\rho t}V(x)\) and if we substitute the derivatives in the HJB equation for the simplest problem, we get the new HJB.

**Observations:**

- if we determine the policy function \(u^* = h(x)\) and substitute in the HJB equation, we see that the new HJB equation is a non-linear ODE

\[
\rho V(x) = f(x, h(x)) + V'(x)g(x, h(x)).
\]

- Differently from the solution from the Pontriyagin’s principle, the HJB defines a recursion over \(x\). Intuitively it generates a rule which says: if we observe the state \(x\) the optimal policy is \(h(x)\) in such a way that the initial value problem should be equal to the present value of the variation of the state.

- It is still very rare to find explicit solutions for \(V(x)\). There is a literature on how to compute it numerically, which is related to the numerical solution of ODE’s and not with approximating value functions as in the discrete time case.

### 3.1.3 Bibliography

3.2 Applications

3.2.1 The cake eating problem

The problem is to find the optimal flows of cake munching $C^* = [C^*(t)]_{t \in [0,T]}$ and of the size of the cake $W^* = [W^*(t)]_{t \in [0,T]}$ such that

$$\max_C \int_0^T \ln(C(t))e^{-\rho t} dt, \text{ subject to } \dot{W} = -C, \ t \in (0, T), \ W(0) = \phi W(T) = 0$$

(3.2)

where $\phi > 0$ is given.

Observation: The problem can be equivalently written as a calculus of variations problem,

$$\max_W \int_0^T \ln(-\dot{W}(t))e^{-\rho t} dt, \text{ subject to } W(0) = \phi W(T) = 0$$

Consider again problem (3.2). Now, we want to solve it by using the principle of the dynamic programming. In order to do it, we have to determine the value function $V = V(t, W)$ which solves the HJB equation

$$-\frac{\partial V}{\partial t} = \max_C \left\{ C e^{-\rho t} \ln(C) - C \frac{\partial V}{\partial W} \right\}$$

The optimal policy for consumption is

$$C^*(t) = e^{-\rho t} \left( \frac{\partial V}{\partial W} \right)^{-1}$$

If we substitute back into the HJB equation we get the partial differential equation

$$-e^{\rho t} \frac{\partial V}{\partial t} = \ln \left[ e^{-\rho t} \left( \frac{\partial V}{\partial W} \right)^{-1} \right] - 1$$

To solve it, let us use the method of undetermined coefficients by conjecturing that the solution is of the type

$$V(t, W) = e^{-\rho t}(a + b \ln W)$$
where $a$ and $b$ are constants to be determined, if our conjecture is right. With this function, the HJB equation comes

$$\rho(a + b \ln W) = \ln(W) - \ln b - 1$$

if we set $b = 1/\rho$ we eliminate the term in $\ln W$ and get

$$a = -(1 - \ln(\rho))/\rho.$$ 

Therefore, solution for the HJB equation is

$$V(t, W) = \frac{-1 + \ln(\rho) + \ln W e^{-\rho t}}{\rho}$$

and the optimal policy for consumption is

$$C^*(t) = \rho W(t).$$
3.2.2 The dynamic consumption-saving problem

Assumptions:

- \( T = \mathbb{R}_+ \), i.e., decisions and transactions take place continuously in time;

- deterministic environment: the agents have perfect information over the flow of endowments \( y \equiv [y(t)]_{t \in \mathbb{R}_+} \) and the relevant prices;

- preferences over flows of consumption, \( [c(t)]_{t \in \mathbb{R}_+} \) are evaluated by the intertemporal utility functional
  \[
  V[c] = \int_0^\infty u(c(t))e^{-\rho t}dt
  \]
  which displays impatience (the discount factor \( e^{-\rho t} \in (0, 1) \)), stationarity \((u(\cdot)\) is not directly dependent on time) and time independence and the instantaneous utility function \((u(\cdot))\) is continuous, differentiable, increasing and concave;

- observe that mathematically the intertemporal utility function is in fact a functional, or a generalized function, i.e., a mapping whose argument is a function (not a number as in the case of functions). Therefore, solving the consumption problem means finding an optimal function. In particular, it consists in finding an optimal trajectory for consumption;

- institutional setting: there are spot real and financial markets that are continuously open. The price \( P(t) \) clear the real market instantaneously. There is an asset market in which a single asset is traded which has the price \( S(t) \) and pays a dividend \( V(t) \).

Derivation of the budget constraint:

The consumer chooses the number of assets \( \theta(t) \). If we consider a small increment in time \( h \) and assume that the flow variables are constant in the interval then

\[
S(t + h)\theta(t + h) = \theta(t)S(t) + \theta(t)D(t)h + P(t)(y(t) - c(t))h.
\]
Define \( A(t) = S(t)\theta(t) \) in nominal terms.

The budget constraint is equivalently

\[
A(t + h) - A(t) = i(t)A(t)h + P(t)(y(t) - c(t))h
\]

where \( i(t) = \frac{D(t)}{S(t)} \) is the nominal rate of return. If we divide by \( h \) and take the limit when \( h \to 0 \) then

\[
\lim_{h \to 0} \frac{A(t + h) - A(t)}{h} \equiv \frac{dA(t)}{dt} = i(t)A(t) + P(t)(y(t) - c(t)).
\]

If we define real wealth and the real interest rate as \( a(t) \equiv \frac{A(t)}{P(t)} \) and \( r(t) = i(t) + \frac{\ddot{P}}{P(t)} \), then we get the **instantaneous budget constraint**

\[
\dot{a}(t) = r(t)a(t) + y(t) - c(t)
\]

where we assume that \( a(0) = a_0 \) given.

Define the **human wealth**, in real terms, as

\[
h(t) = \int_t^\infty e^{-\int_t^\tau r(s)ds}y(s)ds
\]

as from the Leibniz’s rule

\[
\dot{h}(t) \equiv \frac{dh(t)}{dt} = r(t) \int_t^\infty e^{-\int_t^\tau r(s)ds}y(s)ds - y(t) = r(t)h(t) - y(t)
\]

then **total wealth** at time \( t \) is

\[
w(t) = a(t) + h(t)
\]

and we may represent the budget constraint as a function of \( w(.) \)

\[
\dot{w} = \dot{a}(t) + \dot{h}(t) = \dot{a}(t) + r(t)w(t) - c(t)
\]

The instantaneous budget constraint should not be confused with the **intertemporal budget constraint**. Assume the solvability condition holds at time \( t = 0 \)

\[
\lim_{t \to -\infty} e^{-\int_0^t r(\tau)d\tau}a(t) = 0.
\]
Then it is equivalent to the following intertemporal budget constraint,

\[ w(0) = \int_0^\infty e^{-\int_0^t r(\tau)d\tau} c(t) dt, \]

the present value of the flow of consumption should be equal to the initial total wealth.

To prove this, solve the instantaneous budget constraint (3.3) to get

\[ a(t) = a(0)e^{\int_0^t r(\tau)d\tau} + \int_0^t e^{\int_0^\tau r(\tau)d\tau} y(s) - c(s) ds \]

multiply by \( e^{-\int_0^t r(\tau)d\tau} \), pass to the limit \( t \to \infty \), apply the solvability condition and use the definition of human wealth.

Therefore the **intertemporal optimization problem for the representative agent** is to find \([c^*(t), w^*(t)]_{t \in \mathbb{R}_+}\) which maximizes

\[ V[c] = \int_0^{+\infty} u(c(t)) e^{-\rho t} dt \]

subject to the instantaneous budget constraint

\[ \dot{w}(t) = r(t)w(t) - c(t) \]

given \( w(0) = w_0 \).

**Solving the consumer problem** using DP.

The HJB equation is

\[ \rho V(w) = \max_c \left\{ u(c) + V'(w)(rw - c) \right\} \]

where \( w = w(t), c = c(t), r = r(t) \).
We assume that the utility function is homogeneous of degree $\eta$. Therefore it has the properties:

\[
\begin{align*}
    u(c) &= c^\eta u(1) \\
    u'(c) &= c^{\eta-1} u'(1)
\end{align*}
\]

The optimality condition is:

\[
u'(c^*) = V'(w)\]

then

\[
c^* = \left( \frac{V'(w)}{u'(1)} \right)^{\frac{1}{\eta-1}}
\]

substituting in the HJB equation we get the ODE, defined on $V(w)$,

\[
\rho V(w) = r w V'(w) + (u(1) - u'(1)) \left( \frac{V'(w)}{u'(1)} \right)^{\frac{\eta}{\eta-1}}
\]

In order to solve it, we guess that its solution is of the form:

\[
V(w) = B w^\eta
\]

if we substitute in the HJB equation, we get

\[
\rho B w^\eta = \eta r B w^\eta + (u(1) - u'(1)) \left( \frac{\eta B w}{u'(1)} \right)^{\eta}.
\]

Then we can eliminate the term in $w^\eta$ and solve for $B$ to get

\[
B = \left[ \left( \frac{u(1) - u'(1)}{\rho - \eta r} \right) \left( \frac{\eta}{u'(1)} \right) \right]^{\frac{1}{1-\eta}}.
\]

Then, as $B$ is a function of $r = r(t)$, we determine explicitly the value function

\[
V(w(t)) = B w(t)^\eta = \left[ \left( \frac{u(1) - u'(1)}{\rho - \eta r(t)} \right) \left( \frac{\eta}{u'(1)} \right) \right]^{\frac{1}{1-\eta}} w(t)^\eta
\]

as a function of total wealth.
**Observation**: this is one known case in which we can solve explicitly the HJB equation as it is a linear function on the state variable, \( w \) and the objective function \( u(c) \) is homogeneous.

The optimal policy function can also be determined explicitly

\[
c^*(t) = \left( \frac{\eta B(t)}{u'(1)} \right)^{\frac{1}{\eta-1}} w(t) \equiv \pi(t)w(t) \tag{3.4}
\]

as it sets the control as a function of the state variable, and not as depending on the path of the co-state and state variables as in the Pontryagin’s case, sometimes this solution is called **robust feedback control**.

Substituting in the budget constraint, we get the optimal wealth accumulation

\[
w^*(t) = w_0 e^{\int_0^t r(s)-\pi(s)ds} \tag{3.5}
\]

which is a solution of

\[
\dot{w}^* = r(t)w^*(t) - c^*(t) = (r(t) - \pi(t))w^*(t)
\]

Conclusion: the optimal paths for consumption and wealth accumulation \((c^*(t), w^*(t))\) are given by equations (3.4) and (3.5) for any \( t \in \mathbb{R}_+ \).

### 3.2.3 The Ramsey model

This is a problem for a centralized planner which chooses the optimal consumption flow \( c(t) \) in order to maximize the intertemporal utility functional

\[
\max V[c] = \int_0^\infty u(c)e^{-\rho t}dt
\]

where \( \rho > 0 \) subject to

\[
\dot{k}(t) = f(k(t)) - c(t) \\
k(0) = k_0 \text{ given}
\]
The HJB equation is
\[ \rho V(k) = \max_c \{ u(c) + V'(k)(f(k) - c) \} \]

**Benchmark assumptions:** \( u(c) = \frac{c^{1-\sigma}}{1-\sigma} \) where \( \sigma > 0 \) and \( f(k) = Ak^\alpha \) where \( 0 < \alpha < 1 \).

The HJB equation is
\[ \rho V(k) = \max_c \left\{ \frac{c^{1-\sigma}}{1-\sigma} + V'(k)(Ak^\alpha - c) \right\} \]  
(3.6)

the optimality condition is
\[ c^* = \left( V'(k) \right)^{-\frac{1}{\sigma}} \]

after substituting in equation (3.6) we get
\[ \rho V(k) = V'(k) \left( \frac{\sigma}{1-\sigma} V'(k)^{\frac{1}{\sigma}} + Ak^\alpha \right) \]  
(3.7)

In some particular cases we can get explicit solutions, but in general we don’t.

**Particular case:** \( \alpha = \sigma \)  
Equation (3.7) becomes
\[ \rho V(k) = V'(k) \left( \frac{\sigma}{1-\sigma} V'(k)^{-\frac{1}{\sigma}} + Ak^\sigma \right) \]  
(3.8)

Let us conjecture that the solution is
\[ V(k) = B_0 + B_1 k^{1-\sigma} \]

where \( B_0 \) and \( B_1 \) are undetermined coefficients. Then \( V'(k) = B_1(1 - \sigma)k^{-\sigma} \).

If we substitute in equation (3.8) we get
\[ \rho(B_0 + B_1 k^{1-\sigma}) = B_1 \left[ \sigma ((1 - \sigma)B_1)^{-\frac{1}{\sigma}} k^{1-\sigma} + (1 - \sigma)A \right] \]  
(3.9)
Equation (3.9) is true only if

\[ B_0 = \frac{A(1 - \sigma)}{\rho} B_1 \]  
(3.10)

\[ B_1 = \left( \frac{1}{1 - \sigma} \right) \left( \frac{\sigma}{\rho} \right)^{\sigma}. \]  
(3.11)

Then the following function is indeed a solution of the HJB equation in this particular case

\[ V(k) = \left( \frac{\sigma}{\rho} \right)^{\sigma} \left( \frac{A}{\rho} + \frac{1}{1 - \sigma} k^{1-\sigma} \right) \]

The optimal policy function is

\[ c = h(k) = \frac{\rho}{\sigma} k \]

We can determine the optimal flow of consumption and capital \((c^*(t), k^*(t))\) by substituting \(c(t) = \frac{\rho}{\sigma} k(t)\) in the admissibility conditions to get the ODE

\[ \dot{k}^* = Ak^*(t)^{\alpha} - \frac{\rho}{\sigma} k^*(t) \]

for given \(k(0) = k_0\). This a Bernoulli ODE which has an explicit solution as

\[ k^*(t) = \left[ \frac{A\rho}{\sigma} + \left( \frac{1}{k_0^{1-\sigma}} - \frac{A\rho}{\sigma} \right) e^{-(1-\sigma)\rho t} \right]^{1/(1-\sigma)}, \quad t = 0, \ldots, \infty \]

and

\[ c^*(t) = \frac{\rho}{\sigma} k^*(t), \quad t = 0, \ldots, \infty. \]
Part II

Stochastic Dynamic Programming
Chapter 4

Discrete Time

4.1 A short introduction to stochastic processes

4.1.1 Filtrations

We assume that the underlying information is given by a filtration \( \mathcal{F} = \{ \mathcal{F}_0, \mathcal{F}_1, \ldots, \mathcal{F}_T \} \) drawn from a probability space \((\Omega, \mathcal{F}, P)\). The basic magnitudes of the problem are stochastic processes, \( X^t \equiv \{ X_\tau : \tau = 0, 1, \ldots, T \} \) where \( X_t = X(w_t) \) is a random variable where \( w_t \in \mathcal{F}_t \). The information available at period \( t \in \mathbb{T} \) will be represented by the \( \sigma \)-algebra \( \mathcal{F}_t \subset \mathcal{F} \).

A simple example is given by the following binomial information tree

The primitive probability space The event space \( \Omega \) is the set of all elementary events. It can be continuous or discrete, and have a finite or an infinite number of elements. In the following we will assume that

\[
\Omega = \{ \omega_1, \ldots, \omega_s, \ldots, \omega_N \}
\]

Then \( N = \dim(\Omega) \) is the number of possible states of nature.
The $\sigma$-algebra, $\mathcal{F}$ is the set of all subsets of $\Omega$, built by operations of union, and includes the empty set (no event). $P$ is a probability measure over $\mathcal{F}$: that is, for any $A \in \mathcal{F}$, $0 \leq P(A) \leq 1$.

The information through time The information available at time $t \in \mathbb{T}$ will be represented by $\mathcal{F}_t$ and by a filtration, for the sequence of periods $t = 0, 1, \ldots, T$, $\mathbb{F} = \{\mathcal{F}_t\}_{t=0}^T$.

Definition 5. A filtration is a sequence of $\sigma$-algebras $\{\mathcal{F}_t\}$

$$\mathbb{F} = \{\mathcal{F}_0, \mathcal{F}_1, \ldots, \mathcal{F}_T\}.$$  

A filtration is non-anticipating if

- $\mathcal{F}_0 = \{\emptyset, \Omega\}$,
- $\mathcal{F}_s \subseteq \mathcal{F}_t$, if $s \leq t$,  

![Figure 4.1: Information tree](image)
• $\mathcal{F}_T = \mathcal{F}$.

**Intuition:**
(1) Initially we have no information (besides knowing that an event is observable or not);
(2) the information increases with time;
(3) at the terminal moment we not only observe the true state of nature, but also know the past history.

**Example** Taking the example of the binomial tree, we have:

\[
\begin{align*}
\mathcal{F}_0 &= \{\emptyset, \Omega\}, \\
\mathcal{F}_1 &= \{\emptyset, \Omega, \{\omega_1, \omega_2, \omega_3, \omega_4\}, \{\omega_5, \omega_6, \omega_7, \omega_8\}\}, \\
\mathcal{F}_2 &= \{\emptyset, \Omega, \{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}, \{\omega_5, \omega_6\}, \{\omega_7, \omega_8\}, \\
&\quad \{\omega_1, \omega_2, \omega_3, \omega_4\}, \{\omega_5, \omega_6, \omega_7, \omega_8\}\} \\
\mathcal{F}_3 &= \{\emptyset, \Omega, \{\omega_1\}, \{\omega_2\}, \{\omega_3\}, \{\omega_4\}, \{\omega_5\}, \{\omega_6\}, \{\omega_7\}, \{\omega_8\}, \\
&\quad \{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}, \{\omega_5, \omega_6\}, \{\omega_7, \omega_8\}, \\
&\quad \{\omega_1, \omega_2, \omega_3, \omega_4\}, \{\omega_5, \omega_6, \omega_7, \omega_8\}\}
\end{align*}
\]

At any moment $0 \leq t \leq T$, the information available is represented by an **history**

\[w^t = \{w_t, w_{t-1}, \ldots, w_0\}\]

where $w_t \in \mathcal{F}_t$ is the **elementary** component of $\mathcal{F}_t$, that is

\[w_t = (w_{t,1}, \ldots, w_{t,N_t}),\]

where $N_t$ is the number of elements of the finer partition of $\Omega$ belonging to $\mathcal{F}_t$ at time $t$. Therefore, $N_T = N$. Every subset $\mathcal{F}_t$ is composed by a given number of elementary
subsets of $\mathcal{F}$ in every period, denoted by $N_t$, plus subsets formed by all unions of those elementary subsets. However, given the non-anticipating structure of information, the union of $w_t$ components belong to $\mathcal{F}_s$ with $s < t$.

In the previous example, we see that

\[
\begin{align*}
\mathcal{F}_0 &= \{\emptyset, w_0\}, \\
\mathcal{F}_1 &= \{\emptyset, w_0, w_{11}, w_{12}\}, \\
\mathcal{F}_2 &= \{\emptyset, w_0, w_{21}, w_{22}, w_{23}, w_{24}, w_{11}, w_{12}\} \\
\mathcal{F}_3 &= \{\emptyset, w_0, w_{31}, w_{32}, w_{33}, w_{34}, w_{35}, w_{36}, w_{37}, w_{38}, w_{21}, w_{22}, w_{23}, w_{24}, w_{11}, w_{12}\}
\end{align*}
\]

Then $\mathcal{F}_t$ is the set of all the histories up until time $t$,

\[
\mathcal{F}_t = \{\{w_\tau\}_{\tau=0}^t : w_\tau \in \mathcal{P}_\tau, 0 \leq \tau \leq t\}
\]

and $\mathbb{F}$ is the set off all histories.

### 4.1.2 Probabilities over filtrations

We should distinguish probabilities:

- for events $y$, occurring in a moment in time, $t$, $P(w_t = y)$ from probabilities associated to sequences of events $w^t = y^t$, that is $P(\{w_0 = y_0, w_1 = y_1, \ldots, w_t = y_t\})$ or $P(w^t = y^t)$;

- with information taken at time $t = 0$, i.e. $\Omega$; or associated to a particular history $w^t \in \mathcal{F}_t$.

From the last perspective, the elements of $\mathcal{F}_t$ have associated two types of probabilities, depending on the information set available at the time of their determination: unconditional probabilities $\pi^0_t = P(w_t|\mathcal{F}_0)$, where $\pi^0_t = (\pi^0_{t,1}, \ldots, \pi^0_{t,N_t})$, such that $\sum_{s=1}^{N_t} \pi^0_{t,s} = 1$, and conditional probabilities, $\pi^\tau_t$ where $\tau < t$ verify $\pi^\tau_t = \pi^0_t \pi^1_t \ldots \pi^{t-1}_t$. 
Unconditional probabilities, $\pi_t^0(y)$ denotes the probability that the event $y$ occurs at time $t$, assuming the information available at time $t = 0$. If we consider the information available at time $t$ the probability of $w_t = y$, at time $t$, is
\[
\pi_t^0(y) = P(w_t = y) = P(w_t = y | \mathcal{F}_0)
\]
where $w_t \subset \mathcal{F}_t$. As a probability, we have
\[
0 \leq \pi_t^0(.) \leq 1.
\]
From the properties of $\mathcal{P}_t \subset \mathcal{F}_t$, as the finer partition of $\mathcal{F}_t$, we readily see that
\[
\pi_t^0(\mathcal{P}_t) = P(\cup_{s=1}^{N_t} w_{t,s}) = \sum_{s=1}^{N_t} P(w_t = w_{t,s}) = \sum_{s=1}^{N_t} \pi_t^0(w_{t,s}) = 1
\]
We denote by $\pi_t^0(w^t)$ the probability that history $w^t \in \mathcal{F}_t$ occurs. In this case, we have $\pi_T^0(\mathcal{F}_T) = 1$.

Conditional probabilities, $\pi_s^t(y)$ denotes the probability that the event $y$ occurs at time $t$ conditional on the information available at time $s < t$
\[
\pi_s^t(y) = P(w_t = y | \mathcal{F}_s), \ s < t
\]
where $w_t \in \mathcal{P}_t \subset \mathcal{F}_t$ such that
\[
0 \leq \pi_s^t(.) \leq 1.
\]
In particular, the conditional probability that we will have event $y_t$ at time $t$, given a sequence sequence of events $y_{t-1}, \ldots, y_0$ from time $t = 0$ until time $t - 1$ is denoted as
\[
\pi_{t-1}^t(y) = P(w_t = y_t | w_{t-1} = y_{t-1}, \ldots w_0 = w_0),
\]
In order to understand the meaning of the conditional probability, consider a particular case in which it is conditional on the information for a particular previous moment $s < t$. 
Let $\pi_t^s (y|z)$ be the probability that event $y$ occurs at time $t$ given that event $z$ has occurred at time $s < t$,

$$\pi_t^s (y|z) = P(w_t = y|w_s = z)$$

Let us denote $\mathcal{P}_t^y$ and $\mathcal{P}_s^z$ the partitions of $\mathcal{P}_t$ and $\mathcal{P}_s$ which contain, respectively, $y$ and $z$. Then, clearly $\pi_t^s (y|z) = 0$ if $\mathcal{P}_t^y \cap \mathcal{P}_s^z = \emptyset$.

Let us assume that there are at $t$, $n_t^s (z)$ subsets of $\mathcal{P}_t$ such that $\cup_{j=1}^{n_t^s (z)} w_{t,s} = z$. Then

$$\pi_t^s (z|z) = P\left( \cup_{j=1}^{n_t^s (z)} w_{t,s} \big| w_s = z \right) = \sum_{j=1}^{n_t^s (z)} P(w_t = w_{t,j} | w_s = z) = \sum_{j=1}^{n_t^s (z)} \pi_t^s (w_{t,j}) = 1$$

Alternatively, if we use the indicator function $1_t^s (z)$

$$1_t^s (z) = \begin{cases} 
1, & \text{if } \mathcal{P}_t^y \subseteq \mathcal{P}_s^z \\
0, & \text{if } \mathcal{P}_t^y \nsubseteq \mathcal{P}_s^z 
\end{cases}$$

we can write

$$\pi_t^s (z|z) = \sum_{j=1}^{N_t} P(w_t = w_{t,j} | w_s = z) 1_t^s (z) = \sum_{j=1}^{N_t} \pi_t^s (w_{t,j}) 1_t^s (z) = 1$$

This means that $\pi_t^s (y|z)$ is a probability measure starting from a particular node $w_s = z$.

There is a relationship between unconditional and conditional probabilities. Consider the unconditional probabilities, $\pi_0^t (y) = P(w_t = y)$ and $\pi_0^{t-1} (z) = P(w_{t-1} = z)$. We assume that the following relationship between unconditional and conditional probabilities hold $^1$

$$\pi_0^t (y) = \pi_{t-1}^t (y|z) \pi_0^{t-1} (z).$$

For sequences of events we have the following relationship between conditional and unconditional probabilities, if we take information up to time $s = t - 1$

$$P(w_t = y_t | w_{t-1} = y_{t-1}, \ldots, w_0) = \frac{P(w_t = y_t, w_{t-1} = y_{t-1}, \ldots, w_0)}{P(w_{t-1} = y_{t-1}, \ldots, w_0)}$$

$^1$This is a consequence of the Bayes’ rule.
or

\[ \pi_{t-1}^t(y_t | y_{t-1}) = \pi_0^t(y_t) / \pi_0^{t-1}(y_{t-1}) \]

Therefore, if we consider a sequence of events starting from \( t = 0 \), \( \{w_\tau\}_{\tau=0}^t \), we have associated a sequence of conditional probabilities, or transition probabilities

\[ \{\pi_1^0, \pi_1^1, \ldots, \pi_{t-1}^t\} \]

where

\[ \pi_0^t = \prod_{s=1}^{t} \pi_{s-1}^t \]

### 4.1.3 Stochastic processes

**Definition 6.** A stochastic process is a function \( X : \mathbb{T} \times \Omega \rightarrow \mathbb{R} \). For every \( \omega \in \Omega \) the mapping \( t \mapsto X_t(\omega) \) defines a trajectory and for every \( t \in \mathbb{T} \) the mapping \( \omega \mapsto X(t, \omega) \) defines a random variable.

A common representation is the sequence \( X^t = \{X_0, X_1, \ldots, X_t\} = \{X_s\}_{s=0}^t \), which is a sequence of random variables.

**Definition 7.** The stochastic process \( X^T \) is an adapted process as regards the filtration \( \mathbb{F} = \{\mathcal{F}_t : t = 0, 1, \ldots, T\} \) if the random variable \( X_t \) is measurable as regards \( \mathcal{F}_t \), for all \( t \in \mathbb{T} \). That is

\[ X_t = X(w_t) \quad w_t \in \mathcal{F}_t \]

that is \( X_t = (X_1, \ldots X_{N_t}) = (X(w_1), \ldots X(w_{N_t})) \)
Example: Again, in the previous example, a stochastic process adapted to the filtration, has the following possible realizations

\[
X_0 = X(w_0) = \begin{cases} 
  x_0 = 1, & w_0 = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6, \omega_7, \omega_8\}, \\
  x_1 = 1.5, & w_1 = \{\omega_1, \omega_2, \omega_3, \omega_4\} \\
  x_2 = 0.5, & w_2 = \{\omega_5, \omega_6, \omega_7, \omega_8\} \\
  x_3 = 2, & w_2 = \{\omega_1, \omega_2\} \\
  x_2 = 1.2, & w_2 = \{\omega_3, \omega_4\} \\
  x_3 = 0.9, & w_2 = \{\omega_5, \omega_6\} \\
  x_2 = 0.3, & w_2 = \{\omega_7, \omega_8\} \\
  x_3 = 4, & w_3 = \{\omega_1\} \\
  x_3 = 3, & w_3 = \{\omega_2\} \\
  x_3 = 2.5, & w_3 = \{\omega_3\} \\
  x_3 = 2, & w_3 = \{\omega_4\} \\
  x_3 = 1.5, & w_3 = \{\omega_5\} \\
  x_3 = 1, & w_3 = \{\omega_6\} \\
  x_3 = 0.5, & w_3 = \{\omega_7\} \\
  x_3 = 0.125, & w_3 = \{\omega_8\} \\
\end{cases}
\]

Therefore, depending on the particular history, \(\{w_t\}_{t=0}^3\) we have a particular realization \(\{X_t\}_{t=0}^3\), that is a sequence, for example \(\{1, 0.5, 0.9, 1\}\), if the process is adapted. \(\square\)

Definition 8. The stochastic process \(X^T\) is a **predictable process** as regards the filtration \(\mathbb{F} = \{\mathcal{F}_t : t = 0, 1, \ldots, T\}\) if \(X_t\) is a random variable which is measurable as regards \(\mathcal{F}_{t-1}\), for every \(t \in \mathbb{T}\). That is, for time \(0 \leq t \leq T\)

\[
X_t = X(w_{t-1}) = (X_{t1}, \ldots, X_{tN_t-1})
\]
Observation: As $F_{t-1} \subseteq F_t$ then the predictable processes are also adapted as regards the filtration $\mathbb{F}$.

As the underlying space $(\Omega, \mathcal{F}, P)$ is a probability space, there is an associated probability of following a particular sequence $\{x_0, \ldots, x_t\}^2$

$$P(X_t = x_t, X_{t-1} = x_{t-1}, \ldots, X_0 = x_0), \ t \in [0, T].$$

We can write, the conditional probability for $t + 1$

$$P(X_{t+1} = x_{t+1} | X_t = x_t, X_{t-1} = x_{t-1}, \ldots, X_0 = x_0) = \frac{P(X_{t+1} = x_{t+1}, X_t = x_t, X_{t-1} = x_{t-1}, \ldots, X_0 = x_0)}{P(X_t = x_t, X_{t-1} = x_{t-1}, \ldots, X_0 = x_0)} \quad (4.1)$$

We will characterize it next through its conditional and unconditional moments.

For a stochastic process, the **unconditional mathematical expectation** of $X_t$ is

$$E_0(X_t) = E(X_t | \mathcal{F}_0) = \sum_{s=1}^{N_t} P(X_t = x_{t,s}) x_{t,s} = \sum_{s=1}^{N_t} \pi^t_0(x_{t,s}) x_{t,s}$$

where $x_{t,s} = X(w_t = w_{t,s})$, and the **unconditional variance** of $X_t$ is

$$V_0(X_t) = V(X_t | \mathcal{F}_0) = E_0[(X_t - E_0(X_t))^2] = \sum_{s=1}^{N_t} \pi^t_0(x_{t,s})(x_{t,s} - E_0(X_t))^2.$$

The **conditional mathematical expectation** of $X_t$ as regards $\mathcal{F}_s$ with $s \leq t$ is denoted by

$$E_s(X_t) = E(X_t | \mathcal{F}_s).$$

Using our previous notation, we immediatly see that $E_s(X_t)$ is a random variable, measurable as regards $\mathcal{F}_s$, that is

$$E_s(X_t) = (E_{s,1}(X_t), \ldots, E_{s,N_s}(X_t)).$$

\(^2\text{Off course the same definitions apply for any subsequence } t_0, \ldots, t_n.\)
Again, we can consider the case in which the expectation is taken relative to a given history \( X^s = Y^s \) that is to a particular path \( \{x_s = y_s, x_{s-1} = y_{s-1}, \ldots, x_0 = y_0\} \),

\[
E_{s,i}(X_t) = \sum_{j=0}^{N_s} P(X_t = x_{t,j} | Y^s)X_{t,j} = \sum_{j=0}^{N_s} \pi^t_s(X_{t,j} | Y^s)X_{t,j}, \ i = 1, \ldots, N_{t-1}
\]
or relative to a given value of the process at time \( s, X_s \)

\[
E_{s,i}(X_t) = \sum_{j=0}^{N_s} P(X_t = x_{t,j} | X_s = x_{s,i})X_{t,j} = \sum_{j=0}^{N_s} \pi^t_s(X_{t,j} | X_s, i)X_{t,j}, \ i = 1, \ldots, N_{t-1}
\]

Properties of conditional expectation

- \( E_t(X) \geq 0 \) if \( X \geq 0 \);
- \( E_t(aX + bY) = aE_t(X) + bE_t(Y) \) for \( a \) and \( b \) constant;
- \( E_0(X) = E(X) = \sum_{s=1}^N \pi_0(w_s)X(w_s) \);
- \( E_t(1) = 1 \) for any \( t \geq 0 \);
- **law of iterated expectations**: given \( \mathcal{F}_s \subseteq \mathcal{F}_t \), for \( s \leq t \) then

\[
E(X \mid \mathcal{F}_s) = E(E(X \mid \mathcal{F}_t) \mid \mathcal{F}_s),
\]

or, equivalently,

\[
E_s(E_t(X)) = E_s(X);
\]
- if \( Y \) measurable as regards \( \mathcal{F}_t \) then \( E(Y \mid \mathcal{F}_t) = Y \);
- if \( Y \) is independent as regards \( \mathcal{F}_t \) then \( E(Y \mid \mathcal{F}_t) = E(Y) \), that is \( E_t(Y) = E_0(Y) \);
- if \( Y \) is measurable as regards \( \mathcal{F}_t \) then \( E(YX \mid \mathcal{F}_t) = YE(X \mid \mathcal{F}_t) \).
4.1.4 Some important processes

**Stationary process**: a stochastic process \( \{X_t, t \in \mathbb{T}\} \) is stationary if the joint probability is invariant to time shifts

\[
P(X_{t+h} = x_{t+h}, X_{t+h-1} = x_{t+h-1} \ldots X_{s+h} = x_{s+h}) = P(X_t = x_t, X_{t-1} = x_{t-1} \ldots X_s = x_s)
\]

For example, a sequence of independent and identically distributed random variables generates a stationary process.

**Example**: A random walk, \( X \), is a process such that \( X_0 = 0 \) and \( X_{t+1} - X_t \), for \( t = 0, 1, \ldots, T - 1 \) are i.i.d. \( X \) is both a stationary process and a martingale.

**Processes with independent variation**: a process \( \{X_t, t \in \mathbb{T}\} \) has independent variations if the random variation between any two moments \( X_{t_j} - X_{t_{j-1}} \) is independent from any other sequence of time instants.

**Wiener process**

The Wiener process is an example of a stationary stochastic process with independent variations (and continuous sample paths) \( W^t = \{W_t, t \in [0, T]\} \) such that

\[
W_0 = 0, \ E_0(W_t) = 0, \ V_0(W_t - W_s) = t - s,
\]

for any pair \( t, s \in [0, T] \).

**Markov processes and Markov chains**

Markov processes have the **Markov property**:

\[
P(X_{t+h} = x_{t+h}|X^t = x^t) = P(X_{t+h} = x_{t+h}|X_t = x_t)
\]
where $X^t = x^t$ denotes $\{X_t = x_t, X_{t-1} = x_{t-1}, \ldots, X_0 = x_0\}$ and $P(X_{t+h} = x_{t+h}|X^t = x^t)$ is called a transition probability.

That is, the conditional probability of any state in the future, conditional on the past history is only dependent on the present state of the process. In other words, the only relevant probabilities are transition probabilities. The sequences of transition probabilities are independent of the

If we assume that we have a finite number of states, that is $X_t$ can only take a finite number of values

$$\mathcal{Y} = \{y_1, \ldots, y_M\}$$

then we have a (discrete-time) Markov chain. Then, the transition probability can be denoted as

$$\pi_i^j(n) = P(X_{t_{n+1}} = y_j|X_{t_n} = y_i)$$

the transition probability from state $y_i$ to state $y_j$ at time $t_n$. Obviously,

$$0 \leq \pi_i^j(n) \leq 1, \quad \sum_{j=1}^{M} \pi_i^j(n) = 1$$

for any $n$.

The transitional probability $\pi_i^j(n)$ is conditional. We have the unconditional probability

$$\pi_i(n) = P(X_{t_n} = y_i)$$

and the recurrence relationship between conditional and unconditional probabilities

$$\pi_j(n + 1) = \sum_{j=1}^{M} \pi_i^j(n)\pi_i(n)$$

determining the unconditional probability that the Markov chain will be in state $y_j$ at $n+1$. 
We can determine the vector of probabilities for all states in $\mathcal{Y}$ through the transition probability matrix

$$P(n + 1) = \pi(n)P(n)$$

where

$$P(n) = \begin{pmatrix} \pi_1 \\ \vdots \\ \pi_M \end{pmatrix}, \quad \pi = \begin{pmatrix} \pi_1^1 & \ldots & \pi_M^1 \\ \vdots & \ddots & \vdots \\ \pi_1^M & \ldots & \pi_M^M \end{pmatrix}$$
4.2 Stochastic Dynamic Programming

From the set of all feasible $\mathcal{F}_t$-adapted stochastic processes $\{x_t\}_{t=0}^T \{u_t\}_{t=0}^{T-1}$ where $x_t = x(w_t)$ and $u_t = u(w_t)$, choose a contingent plan $\{x^*_t\}_{t=0}^T \{u^*_t\}_{t=0}^{T-1}$ that

$$ \max_{\{u_t\}_{t=0}^{T-1}} E_0 \left[ \sum_{t=0}^{T} \beta^t f(u_t, x_t) \right] = \max_{\{u_t\}_{t=0}^{T-1}} E \left[ \sum_{t=0}^{T} \beta^t f(u_t, x_t) \mid \mathcal{F}_0 \right] $$

subject to the random sequence of constraints

$$ x_1 = g(x_0, u_0, w^1) $$

$$ \ldots $$

$$ x_{t+1} = g(x_t, u_t, w^{t+1}), \ t = 1, \ldots, T - 2 $$

$$ \ldots $$

$$ x_T = g(x_{T-1}, u_{T-1}, w^T) $$

where $x_0$ is given and $w^t = \{w_t, \ldots, w_0\}$ is a $\mathcal{F}_t$-adapted process representing the (exogenous) uncertainty affecting the agent decision.

**Timings of information availability and decisions:** at the beginning of a period $t$, $x_t$ and $u_t$ are known but the value of $x_{t+1}$, at the end of period $t$ is unknown, because it conditional on the value of $w^{t+1}$ which is not known at the moment of the decision over $u_t$. As the decision on $u_t$ depends on $x_t$, which has been influenced by past decisions, and on future decisions over $u$, because of the intertemporal nature of the decision criterium, then the decision over $u_t$ involves a conditional plan.

Intuitively, we may say that in this case the decision over the control variable occurs at the beginning of period $t$.

Let us call $u_t^*$ the optimal control at time $t$. Then $\{u_t^*\}_{t=0}^T$ represents an optimal contingent plan, i.e., a planned sequence of decisions conditional on the sequence of states of nature (or events).
Again, define the value function at time $\tau \in \{0, \ldots, T - 1\}$ as

$$ V_{\tau}(x_{\tau}) = E_{\tau} \left[ \sum_{t=\tau}^{T-1} \beta^{t-\tau} f(u_{t}^{*}, x_{t}) \right] = \max_{\{u_{t}\}_{t=\tau}} E_{\tau} \left[ \sum_{t=\tau}^{T-1} \beta^{t-\tau} f(u_{t}, x_{t}) \right]. \tag{4.5} $$

Now the value function is the expected value, taking information at time $t = 0$, of the present value of the sum of period utilities until $T - 1$.

At time $t = 0$ we get

$$ V(x_{0}) = E_{0} \left[ \sum_{t=0}^{T} \beta^{t} f(u_{t}^{*}, x_{t}) \right] = \max_{\{u_{t}\}_{t=0}} E_{0} \left[ \sum_{t=0}^{T} \beta^{t} f(u_{t}, x_{t}) \right] = \max_{\{u_{t}\}_{t=0}} f(u_{0}, x_{0}) + \beta \sum_{t=1}^{T-1} \beta^{t-1} f(u_{t}^{*}, x_{t}) = \max_{u_{0}} \left\{ f(u_{0}, x_{0}) + \beta \max_{\{u_{t}\}_{t=1}} E_{0} \left[ \sum_{t=1}^{T-1} \beta^{t-1} f(u_{t}^{*}, x_{t}) \right] \right\} = \max_{u_{0}} \left\{ f(u_{0}, x_{0}) + \beta \max_{\{u_{t}\}_{t=1}} E_{0} \left[ E_{1} \left[ \sum_{t=1}^{T-1} \beta^{t-1} f(u_{t}^{*}, x_{t}) \right] \right] \right\} \right\} $$

if we decompose the sum, and apply the principle of DP and observe that $u_{t}$ and $x_{t}$ are both $\mathcal{F}_{t}$-measurable. In the last step we use the law of iterated expectations.

Therefore

$$ V(x_{0}) = \max_{u_{0}} \{ f(u_{0}, x_{0}) + \beta E_{0}[V(x_{1})] \} $$

where $u_{0}$, $x_{0}$ and $V_{0}$ are $\mathcal{F}_{0}$-adapted and $x_{1} = g(u_{0}, x_{0}, w_{1})$, in $V_{1}$, is $\mathcal{F}_{1}$-adapted.

The same idea can be extended to any $0 \leq t \leq T$. Then we get the stochastic HJB equation

$$ V(x_{t}) = \max_{u_{t}} \{ f(u_{t}, x_{t}) + \beta E_{t}[V(x_{t+1})] \} $$

or

$$ V(x_{t}) = \max_{u_{t}} \{ f(u_{t}, x_{t}) + \beta E[V(x_{t+1}) | \mathcal{F}_{t}] \}. $$
The optimal policy function, can be obtained from

\[
\frac{\partial}{\partial u_t} \left( f(u_t, x_t) + \beta E_t[V(x_{t+1})] \right) = 0
\]

Under boundness conditions, for \( V_t \), the same HJB equation holds formally to the \( T \to \infty \) case.

Observe that \( \{V(x_t)\}_{t=0}^{\infty} \) is a \( \mathcal{F}_t \)-adapted stochastic process and the operator \( E_t(\cdot) \) is a probability measure conditional on the information available at \( t \) (represented by \( \mathcal{F}_t \)).

If \( \{x\}_{t=0}^{T} \) follows a \( k \)-state Markov process then the HJB equation can be written as

\[
V(x_t) = \max_{u_t} \left\{ f(u_t, x_t) + \beta \sum_{s=1}^{k} \pi(s) V(x_{t+1}(s)) \right\}
\]
4.3 Applications

4.3.1 The representative consumer

Assumptions:

- there are $K$ short run financial assets which have a price $S^j_t$, $j = 1, \ldots, K$ at time $t$ that entitle to a contingent payoff $D^j_{t+1}$ at time $t+1$;

- the value of the portfolio at the end of period $t$ is $\sum_{j=1}^{K} \theta^j_{t+1} S^j_t$, and its conditional payoff, at the beginning of period $t+1$ is $\sum_{j=1}^{K} \theta^j_{t+1} (S^j_{t+1} + V^j_{t+1})$, where $\theta_{t+1}$, $S^j_t$ and $V^j_t$ are $\mathcal{F}_t$-measurable,

- the stream of endowments, is $\{y_t\}_{t=0}^T$ where $y_t$ is $\mathcal{F}_t$-measurable;

- $A_0$ may be different from zero.

Budget constraints The consumer faces a (random) sequence of budget constraints, which defines his feasible contingent plans:

- at time $t = 0$
  \[ c_0 + \sum_{j=1}^{K} \theta^j_1 S^j_0 = y_0 + A_0 \]
  where all the components are scalars.

- At time $t = 1$, we have
  \[ c_1 + \sum_{j=1}^{K} \theta^j_2 S^j_1 = y_1 + \sum_{j=1}^{K} \theta^j_1 (S^j_1 + V^j_1) \]
  where $c_1$, $y_1$ and $\theta_2$ are $\mathcal{F}_1$-measurable, that is
  \[ c_1(s) + \sum_{j=1}^{K} \theta^j_2(s) S^j_1(s) = y_1(s) + \sum_{j=1}^{K} \theta^j_1(S^j_1(s) + V^j_1(s)), \ s = 1, \ldots, N_1. \]
• at any period, we have
\[ c_t(s) + \sum_{j=1}^{K} \theta_{t+1}^j(s)S_t^j(s) = y_t(s) + \sum_{j=1}^{K} \theta_t^j(S_t^j(s) + V_t^j(s)), \ s = 1, \ldots, N_t. \]

If we define
\[ A_{t+1}^j = \theta_{t+1}^jS_t^j \]
as the stock of the asset \( j \) at the end of period \( t \) (or at the beginning of period \( t + 1 \)), and
\[ R_t^j = \frac{S_t^j + V_t^j}{S_{t-1}^j} \]
as the return of asset at the end of period \( t \), then we can write the budget constraint at time \( t \) as
\[ c_t(s) + \sum_{j=1}^{K} A_{t+1}^j(s) = y_t(s) + \sum_{j=1}^{K} R_t^j(s)A_t^j, \ s = 1, \ldots, N_t. \]

Observe that \( c_t, y_t, A_{t+1}^j \) and \( R_t \) are \( \mathcal{F}_t \)-adapted. Therefore \( A_t^j \) is \( \mathcal{F}_{t-1} \)-adapted.

The representative consumer chooses a strategy of consumption, represented by the adapted process \( c := \{c_t, t \in \mathbb{T}\} \) and of financial transactions in \( K \) financial assets, represented by the forecastable process \( \theta := \{\theta_t, t \in \mathbb{T}\} \), where \( \theta_t = (\theta_1^t, \ldots, \theta_K^t) \) in order to solve the following problem
\[
\max_{\{c, \theta\}} E_0 \left[ \sum_{t=0}^{\infty} \beta^t u(c_t) \right]
\]
subject to equations (4.8)-(4.9), where \( A(0) = A_0 \) is given and
\[
\lim_{k \to \infty} E_t \left[ \beta^k S_{t+k}^j \right] = 0.
\]

The last condition prevents consumers from playing Ponzi games. It rules out the existence of arbitrage opportunities.
4.3.2 The consumer problem: alternative methods of solution

Calculus of variations approach

**Only one financial asset**  Let us assume that the horizon, \( T \), is finite and that there is only one financial asset. In this case we have the budget constraint

\[
A_{t+1} = y_t - c_t + R_t A_t
\]  \tag{4.6}

If we substitute \( c_t \) from equation (4.6) in intertemporal utility function, we have

\[
\max_{\{c_t\}_{t=0}^T} E_0 \left[ \sum_{t=0}^{T} \beta^t u(c_t) \right] = \max_{\{A_{t+1}\}_{t=0}^T} E_0 \left[ \sum_{t=0}^{T} \beta^t u(y_t + R_t A_t - A_{t+1}) \right]
\]

If we maximize for \( A_{t+1} \), for \( t = 0, \ldots, T - 1 \), the stochastic Lagrange-Euler condition is

\[
\frac{\partial u(c_t)}{\partial A_{t+1}} + \beta E_t \left[ \frac{\partial u(c_{t+1})}{\partial A_{t+1}} \right] = 0, \quad t = 0, \ldots, T - 1
\]
equivalently

\[
u'(c_t) = \beta E_t \left[ u'(c_{t+1}) R_{t+1} \right], \quad t = 0, \ldots, T - 1.
\]

If the information structure follows a Markov chain with \( n \) nodes, then we can write the equilibrium asset pricing equation for node \( n_t \)

\[
u'(c_t(n_t)) = \beta \sum_{s=1}^{n} \pi_s u'(c_{t+1,s}(n_t)) R_{t+1,s}(n_t), \quad n_t = 1, \ldots, n^t, \quad t = 0, \ldots, T - 1.
\]

In order to solve explicitly this equation we would need an explicit probability distribution.

**Several financial assets**  Now, let us assume that there are \( K \) financial assets: in this case we have the budget constraint becomes

\[
\sum_{j=1}^{K} A_{t+1}^j = y_t - c_t + \sum_{j=1}^{K} R_{t}^j A_t^j \tag{4.7}
\]
The value function is now

\[
\max_{\{A^1_{t+1}, \ldots, A^K_{t+1}\}_{t=0}^T} E_0 \left[ \sum_{t=0}^T \beta^t u \left( y_t + \sum_{j=1}^K R^j_t A^j_t - \sum_{j=1}^K A^j_{t+1} \right) \right]
\]

The stochastic Lagrange-Euler condition are

\[
\frac{\partial u(c_t)}{\partial A^j_{t+1}} + \beta E_t \left[ \frac{\partial u(c_{t+1})}{\partial A^j_{t+1}} \right] = 0, \ j = 1, \ldots, K, \ t = 0, \ldots, T - 1
\]

equivalently

\[
u'(c_t) = \beta E_t \left[ u'(c_{t+1}) R^j_{t+1} \right], \ j = 1, \ldots, K, \ t = 0, \ldots, T - 1.
\]

**Dynamic programming approach**

If we look at the budget constraint for period \( t \)

\[
c_t + \sum_{j=1}^K A^j_{t+1} = y_t + \sum_{j=1}^K R^j_t A^j_t
\]

we see that it assumes implicitly a different decision timing than equation (4.3). Then the optimal control problem for this problem is not as in the benchmark case. Therefore we should come up with a different strategy for deriving the HJB equation.

**Ljungqvist and Sargent approach**

Write the sequence of instantaneous budget constraints is

\[
y_t + a_t \geq c_t + \sum_{j=1}^K \theta^j_{t+1} S^j_t, \ t \in [0, \infty) \quad (4.8)
\]

\[
a_{t+1} = \sum_{j=1}^K \theta^j_{t+1} (S^j_{t+1} + V^j_{t+1}), \ t \in [0, \infty)
\]

(4.9)
This representation of the problem allows us to solve it by using dynamic programming (see Ljungqvist and Sargent (2004)).

The Bellman equation is

\[
V(a_t) = \max_{c_t} \{u(c_t) + \beta E_t [V(a_{t+1})]\}.
\]

Observe that, in this approach, \(c_t\) and \(a_t\) are \(\mathcal{F}_t\)-measurable. In our case, we may solve it by determining the optimal transactions strategy

\[
V(a_t) = \max_{\theta^j_t, j=1, \ldots, K} \left\{ \left\{ u(\left[ y_t + a_t - \sum_{j=1}^{K} \theta^j_{t+1} S^j_t \right] + + \beta E_t [V(a_{t+1})(\theta^j_{t+1}, \ldots, \theta^K_{t+1})]) \right\} \right\}
\]

(4.10)

The optimality condition, is

\[
-u'(c_t)S^j_t + \beta E_t [V'(a_{t+1})(S^j_{t+1} + V^j_{t+1})] = 0,
\]

for every asset \(j = 1, \ldots, K\).

In order to simplify this expression, we may apply the Benveniste-Scheinkman formula (see (Ljungqvist and Sargent, 2000, p.237)) by substituting the optimality conditions in equation (4.10) and by differentiating it in order to \(a_t\). Then we get

\[
V'(a_t) = u'(c_t).
\]

Then, again

\[
-u'(c_t)S^j_t + \beta E_t \left[ u'(c_{t+1})(S^j_{t+1} + V^j_{t+1}) \right] = 0,
\]
Alternative dynamic programming approach

Let us assume that there is only one financial asset. If we assume that the value function is a function of the financial wealth, $A_t$, then the Bellman equation becomes

$$V(A_t) = \max_{c_t} E_{t-1} [u(c_t) + \beta V(A_{t+1})]$$

where $A_{t+1} = y_t - c_t + R_tA_t$ is $\mathcal{F}_t$-measurable (observe the location of the expectation operator).

If we maximize for $c_t$ and apply the envelope condition, we get the conditions for optimality

$$u'(c_t) = \beta V'(A_{t+1})$$
$$V'(A_{t+1}) = \beta E_t \left[ V'(A_{t+2}) R_{t+1} \right]$$

and we get the same optimality condition

If we have $K$ financial assets, again $A_t = \sum_{j=1}^{K} A_t^j$ and $R_tA_t = \sum_{j=1}^{K} R_t^j A_t^j$, and the optimality conditions become

$$u'(c_t) = \beta V'(A_{t+1})$$
$$V'(A_{t+1}) \frac{\partial A_{t+1}}{\partial A_{t+1}^j} = \beta E_t \left[ V'(A_{t+2}) \frac{\partial A_{t+2}}{\partial A_{t+1}^j} \right], \ j = 1, \ldots, K$$

The last equation becomes

$$V'(A_{t+1}) = \beta E_t \left[ V'(A_{t+2}) R_{t+1}^j \right], \ j = 1, \ldots, K$$

and we get again the arbitrage condition for asset $j$.

4.3.3 Infinite horizons

Therefore, the optimality conditions may be rewritten as the following intertemporal arbitrage conditions for the representative consumer

$$u'(c_t) S_t^j = \beta E_t \left[ u'(c_{t+1}) (S_{t+1}^j + V_{t+1}^j) \right], \ j = 1, \ldots, K, \ t \in [0, \infty)$$ (4.11)
However, the absence of arbitrage opportunities in the financial market imposes conditions on the asymptotic properties of prices, which imposes conditions on the solution of the consumer’s problem.

Taking equation (4.11) and operation recursively, the consumer chooses an optimal trajectory of consumption such that (remember that the asset prices and the payoffs are given to the consumer)

\[ S^j_t = E_t \left[ \sum_{\tau=1}^{\infty} \beta^\tau \frac{u'(c_{t+\tau})}{u'(c_t)} V^j_{t+\tau} \right], \quad j = 1, \ldots, K, \quad t \in [0, \infty) \]  

(4.12)

In order to prove this note that by repeatedly applying the law of iterated expectations

\[ u'(c_t)S^j_t = \beta E_t \left[ u'(c_{t+1})(S^j_{t+1} + V^j_{t+1}) \right] = \]

\[ \beta E_t \left[ u'(c_{t+1})S^j_{t+1} \right] + \beta E_t \left[ u'(c_{t+1})V^j_{t+1} \right] = \]

\[ \beta E_t \left\{ \beta E_{t+1} \left[ u'(c_{t+2})(S^j_{t+2} + V^j_{t+2}) \right] \right\} + \beta E_t \left[ u'(c_{t+1})V^j_{t+1} \right] \]

repeating the same procedure,

\[ u'(c_t)S^j_t = \beta E_t \left\{ \beta E_{t+1} \left[ u'(c_{t+2})S^j_{t+2} \right] \right\} + \]

\[ + E_t \left\{ \beta u'(c_{t+1})V^j_{t+1} + \beta^2 E_{t+1} \left[ u'(c_{t+2})V^j_{t+2} \right] \right\} = \]

\[ \beta^2 E_t \left[ u'(c_{t+2})S^j_{t+2} \right] + E_t \left[ \sum_{\tau=1}^{2} \beta^\tau u'(c_{t+\tau})V^j_{t+\tau} \right] = \]

\[ \ldots \]

\[ = \beta^k E_t \left[ u'(c_{t+k})S^j_{t+k} \right] + E_t \left[ \sum_{\tau=1}^{k} \beta^\tau u'(c_{t+\tau})V^j_{t+\tau} \right] = \]

\[ \ldots \]

\[ = \lim_{k \to \infty} \beta^k E_t \left[ u'(c_{t+k})S^j_{t+k} \right] = \lim_{k \to \infty} \beta^k \left[ u'(c_{t+k})S^j_{t+k} \right] = 0 \]

The condition for ruling out speculative bubbles,
allows us to get equation (4.12).

References: Ljungqvist and Sargent (2000)
Chapter 5

Continuous time

5.1 Introduction to continuous time stochastic processes

Assume that $T = \mathbb{R}_+$ and that the probability space is $(\Omega, \mathcal{F}, P)$ where $\Omega$ is of infinite dimension.

Let $\mathbb{F} = \{\mathcal{F}_t, t \in T\}$ be a filtration over the probability space $(\Omega, \mathcal{F}, P)$. $(\Omega, \mathcal{F}, \mathbb{F}, P)$ may be called a filtered probability space.

A stochastic process is a flow $X = \{X(t, \omega), t \in T, \omega \in \mathcal{F}_t\}$.

5.1.1 Brownian motions

Definition 9. Brownian motion Assume the probability space $(\Omega, \mathcal{F}, P^x)$, a sequence of sets $F_t \in \mathbb{R}$ and the stochastic process $B = \{B(t), t \in T\}$ such that a sequence of distributions over $B$ are given by

$$
\mathbb{P}^x(B(t_1) \in F_1, \ldots, B(t_k) \in F_k) = \int_{F_1 \times \cdots \times F_k} p(t_1, x_1) \cdots p(t_2 - t_1, x_1, x_2) \cdots p(t_k - t_{k-1}, x_{k-1}, x_k)dx_1dx_2 \cdots dx_k,
$$
where the conditional probabilities are
\[
P(B(t_j) = x_j \mid B(t_i) = x_i) = p(t_j - t_i, x_i, x_j) = (2\pi(t_j - t_i))^{-\frac{1}{2}} e^{-\frac{|x_j - x_i|^2}{2(t_j - t_i)}}.
\]

Then $B$ is a Brownian motion (or Wiener process), starting from the initial state $x$, where $(P^x(B(0) = x) = 1)$.

**Remark** We consider one-dimensional Brownian motions: that is, those for which the trajectories have continuous versions, $B(\omega) : \mathbb{T} \to \mathbb{R}^n$ where $t \mapsto B_t(\omega)$, with $n = 1$.

**Properties of $B$**

1. $B$ is a gaussian process:

   that is, $Z = (B(t_1), \ldots, B(t_k))$ has a normal distribution with mean $M = E^x[Z] = (x, \ldots, x) \in \mathbb{R}^k$ and variance-covariance matrix

   \[
   E^x[(Z_j - M_j)(Z_i - M_i)]_{i,j=1,\ldots,k} = \begin{bmatrix}
   t_1 & t_1 & \ldots & t_1 \\
   t_1 & t_2 & \ldots & t_2 \\
   \vdots & \vdots & \ddots & \vdots \\
   t_1 & t_1 & \ldots & t_k
   \end{bmatrix}
   \]

   and, for any moment $t \geq 0$

   \[
   E^x[B(t)] = x,
   
   E^x[(B(t) - x)^2] = t,
   
   E^x[(B(t) - x)(B(s) - x)] = \min(t, s),
   
   E^x[(B(t) - B(s))^2] = t - s.
   \]
2. $B$ has independent variations:
   
   given a sequence of moments $0 \leq t_1 \leq t_2 \leq \ldots \leq t_k$ and the sequence of variations of a Brownian motion, $B_{t_2} - B_{t_1}, \ldots, B_{t_k} - B_{t_{k-1}}$ we have
   
   $$E^x[(B(t_i) - B(t_{i-1})(B(t_j) - B(t_{j-1})) = 0, \ t_i < t_j$$

3. $B$ has continuous versions;

4. $B$ is a stationary process:
   
    that is $B(t + h) - B(t)$, with $h \geq 0$ has the same distribution for any $t \in \mathbb{T}$;

5. $B$ is not differentiable (with probability 1) in the Riemannian sense.

**Observation:** it is very common to consider $B(0) = 0$, that is $x = 0$.

### 5.1.2 Processes and functions of $B$

As with random variables and with stochastic processes over a finite number of periods and states of nature:

(1) if we can define a filtration, we can build a stochastic process or,

(2) given a stochastic process, we may define a filtration.

Observation: for random variables (if we define a measure we can define a random variable, or a random variables may induce measures in a measurable space).

**Definition 10.** (filtration)

$\mathbb{F} = \{\mathcal{F}_t, t \in \mathbb{T}\}$ is a filtration if it verifies:

(1) $\mathcal{F}_s \subset \mathcal{F}_t$ if $s < t$

(2) $\mathcal{F}_t \subset \mathcal{F}$ for any $t \in \mathbb{T}$
Definition 11. (Filtration over a Brownian motion)
Consider a sequence of subsets of \( \mathbb{R} \), \( F_1, F_2, \ldots, F_k \) where \( F_j \subset \mathbb{R} \) and let \( B \) be a Brownian motion of dimension 1. \( \mathcal{F}_t \) is a \( \sigma \)-algebra generated by \( B(s) \) such that \( s \leq t \), if it is the finest partition which contains the subsets of the form
\[
\{ \omega : B_{t_1}(\omega) \in F_1, \ldots, B_{t_k}(\omega) \in F_k \}
\]
if \( t_1, t_2, \ldots, t_k \leq t \).

**Intuition** \( \mathcal{F}_t \) is the set of all the histories of \( B \) up to time \( t \).

Definition 12. (\( \mathcal{F}_t \)-measurable function)
A function \( h(\omega) \) is called a \( \mathcal{F}_t \)-measurable if and only if it can be expressed as the limit of the sum of function of the form
\[
h(t, \omega) = g_1(B(t_1))g_2(B(t_2)) \ldots g_k(B(t_k)), \quad t_1, t_2, \ldots, t_k \leq t.
\]

**Intuition** \( h \) is a function of present and past values of a Brownian motion.

Definition 13. (\( \mathcal{F}_t \)-adapted process)
If \( \mathbb{F} = \{ \mathcal{F}_t, t \in T \} \) is a filtration then the process \( g = \{ g(t, \omega), \quad t \in T, \quad \omega \in \Omega \} \) is called \( \mathcal{F}_t \)-adapted if for every \( t \geq 0 \) the function \( \omega \mapsto g_t(\omega) \) is \( \mathcal{F}_t \)-measurable.

For what is presented next, there are two important types of functions and processes:

Definition 14. (Class N functions)
Let \( f : T \times \Omega \rightarrow \mathbb{R} \). If
1. \( f(t, \omega) \) is \( \mathcal{F}_t \)-adapted;
2. \( E \left[ \int_s^T f(t, \omega)^2 dt \right] < \infty \),
then \( f \in N(s, T) \), is called a class \( N(s, T) \) function.

**Definition 15. (Martingale)**
The stochastic process \( M = \{M(t), t \in \mathbb{T}\} \) defined over \((\Omega, \mathcal{F}, P)\) is a martingale as regards the filtration \( \mathbb{F} \) if

1. \( M(t) \) is \( \mathcal{F}_t \)-measurable, for any \( t \in \mathbb{T} \),
2. \( E[|M(t)|] < \infty \), for any \( t \in \mathbb{T} \),
3. the martingale property holds

\[
E[M(s) | \mathcal{F}_t] = M(t),
\]

for any \( s \geq t \).

### 5.1.3 Itô’s integral

**Definition 16. (Itô’s integral)**
Let \( f \) be a function of class \( N \) and \( B(t) \) a one-dimensional Brownian motion. Then the Itô’s integral is denoted as

\[
I(f, \omega) = \int_s^T f(t, \omega)dB_t(\omega)
\]

If \( f \) is a class \( N \) function, it can be proved that the sequence of elementary functions of class \( N \) \( \phi_n \) where

\[
\int_s^T \phi(t, \omega)dB_t(\omega) = \sum_{j=0}^{\infty} e_j(\omega)[B_{t_j+1}(\omega) - B_{t_j}(\omega)],
\]

verifying \( \lim_{n \to \infty} E[\int_s^T |f - \phi_n|^2 \, dt] = 0 \), such that Itô’s integral is defined as

\[
I(f, \omega) = \int_s^T f(t, \omega)dB_t(\omega) = \lim_{n \to \infty} \int_s^T \phi_n(t, \omega)dB_t(\omega).
\]

**Intuition:** as \( f \) is not differentiable (in the Riemannian sense), we may have several definitions of integral. Itô’s integral approximates the function \( f \) by step functions the \( e_j \).
evaluated at the beginning of the interval \((t_{j+1}, t_j)\). The Stratonovich integral
\[
\int_s^T f(t, \omega) \circ dB_t(\omega)
\]
evaluates in the intermediate point of the intervals.

**Theorem 1.** (Properties of Itô's integral)
Consider two class \(N\) function \(f, g \in N(0, T)\), then
1. \(\int_s^T f dB_t = \int_s^U f dB_t + \int_U^T f dB_t\) for almost every \(\omega\) and for \(0 \leq s < U < T\);
2. \(\int_s^T (cf + g) dB_t = c \int_s^T f dB_t + \int_s^T g dB_t\) for almost all \(\omega\) and for \(c\) constant;
3. \(E\left(\int_s^T f dB_t\right) = 0\);
4. has continuous versions up to time \(t\), that is there is a stochastic process \(V = \{V(t), t \in \mathbb{T}\}\) such that \(P(\int_0^t f dB_t = 1) = 1\) for any \(0 \leq t \leq T\);
5. \(M(t, \omega) = \int_0^t f(s, \omega) dB_s\) is a martingale as regards the filtration \(\mathcal{F}_t\).

### 5.1.4 Stochastic integrals

Up to this point we presented a theory of integration, and implicitly of differentiation. The Itô’s presents a very useful stochastic counterpart of the chain rule of differentiation.

**Definition 17.** (Itô’s process or stochastic integral)

Let \(B_t\) be a one-dimensional Brownian motion over \((\Omega, \mathcal{F}, P)\). Let \(\nu\) be a class \(N\) function (i.e., such that \(P\left(\int_0^t \nu(s, \omega)^2 ds < \infty, \forall t \geq 0\right) = 1\)) and let \(\mu\) be a function of class \(\mathcal{H}_t\) (i.e., such that 
\(P\left(\int_0^t |\mu(s, \omega)| ds < \infty, \forall t \geq 0\right) = 1\)).

Then \(X = \{X(t), t \in \mathbb{T}\}\) where \(X(t)\) has the domain \((\Omega, \mathcal{F}, P)\), is a stochastic integral of dimension one if it is a stochastic process with the following equivalent representations:
1. integral representation

\[ X(t) = X(0) + \int_0^t \mu(s, \omega)ds + \int_0^t \nu(s, \omega)dB_s \]

2. differential representation

\[ dX(t) = u(t, \omega)dt + \nu(t, \omega)dB(t) \]

Lemma 3. (Itô’s lemma)

Let \( X(t) \) be a stochastic integral in its differential representation

\[ dX(t) = \mu dt + \nu dB(t) \]

and let \( g(t, x) \) be a continuous differentiable function as regards its two arguments. Then

\[ Y = \{ Y(t) = g(t, X(t)), t \in \mathbb{T} \} \]

is a stochastic process that verifies

\[ dY(t) = \frac{\partial g}{\partial t}(t, X(t))dt + \frac{\partial g}{\partial x}(t, X(t))dX(t) + \frac{1}{2} \frac{\partial^2 g}{\partial x^2}(t, X(t))(dX(t))^2. \]

We apply the rule: \( dt^2 = dt dB(t) = 0 \) e \( dB(t)^2 = dt \) then

\[ dY(t) = \left( \frac{\partial g}{\partial t}(t, X(t)) + \frac{\partial g}{\partial x}(t, X(t))\mu + \frac{1}{2} \frac{\partial^2 g}{\partial x^2}(t, X(t))\nu^2 \right) dt + \frac{\partial g}{\partial x}(t, X(t))\nu dB(t). \]

Example 1 Let \( X(t) = B(t) \) where \( B \) is a Brownian motion. Which process follows \( Y(t) = (1/2)B(t)^2 \)? If we write \( Y(t) = g(t, x) = (1/2)x^2 \) and apply Itô’s lemma we get

\[ dY(t) = dt \left( \frac{1}{2}B(t)^2 \right) = \]

\[ = \frac{\partial g}{\partial t}dt + \frac{\partial g}{\partial x}dB(t) + \frac{1}{2} \frac{\partial^2 g}{\partial x^2}(dB(t))^2 = \]

\[ = 0 + B(t)dB(t) + \frac{1}{2}dB(t)^2 \]

\[ = B(t)dB(t) + \frac{dt}{2}. \]
or in the integral representation
\[ Y(t) = \int_0^t dY(t) = \frac{1}{2} B(t)^2 = \int_0^t B(s) dB(s) + \frac{t}{2}. \]

**Example 2** Let \( dX(t) = \mu X(t) dt + \sigma X(t) dB(t) \) and let \( Y(t) = \ln(X(t)) \). Find the SDE for \( Y \). Applying the Itô’s lemma

\[
dY(t) = \frac{\partial Y}{\partial X} dX(t) + \frac{1}{2} \frac{\partial^2 Y}{\partial X^2} (dX(t))^2 = \frac{dX(t)}{X(t)} - \frac{1}{2X(t)^2} (dX(t))^2 = \\
= \mu dt + \sigma dB(t) - \frac{\sigma^2}{2} dt
\]

Then
\[
dY(t) = \left( \mu - \frac{\sigma^2}{2} \right) dt + \sigma dB(t)
\]
or, in the integral representation
\[
Y(t) = Y(0) + \int_0^t \left( \mu - \frac{\sigma^2}{2} \right) ds + \int_0^t \sigma dB(s) = \\
= Y(0) + \left( \mu - \frac{\sigma^2}{2} \right) t + \sigma B(t)
\]
if \( B(0) = 0 \).

The process \( X \) is called a **geometric Brownian motion**, as its integral (which is the \( \exp(Y) \) is

\[
X(t) = X(0) e^{\left( \mu - \frac{\sigma^2}{2} \right) t + \sigma B(t)}.
\]

**Example 3:** Let \( Y(t) = e^{aB(t)} \). Find a stochastic integral for \( Y \). From the Itô’s lemma

\[
dY(t) = ae^{aB(t)} dB(t) + \frac{1}{2} a^2 e^{aB(t)} (dB(t))^2 = \\
= \frac{1}{2} a^2 e^{aB(t)} dt + ae^{aB(t)} dB(t)
\]
the integral representation is

\[ Y(t) = Y(0) + \frac{1}{2} a^2 \int_0^t e^{aB(s)} ds + a \int_0^t e^{aB(s)} dB(s) \]

### 5.1.5 Stochastic differential equations

Stochastic differential equation’s theory is a very vast field. Here we will only present some results that we will be useful afterwards.

**Definition 18. (SDE)**

A stochastic differential equation can be defined as

\[ \frac{dX(t)}{dt} = b(t, X(t)) + \sigma(t, X(t))W(t) \]

where \( b(t, x) \in \mathbb{R}, \sigma \in \mathbb{R} \) and \( W(t) \) represents a one-dimensional "noise".

**Definition 19. (SDE: Itô’s interpretation)**

\( X(t) \) satisfies a stochastic differential equation is

\[ dX(t) = b(t, X(t))dt + \sigma(t, X(t))dB(t) \]

or in the integral representation

\[ X(t) = X(0) + \int_0^t b(s, X(s))ds + \int_0^t \sigma(s, X(s))dB(s). \]

How to solve, or study qualitatively, the solution of those equations?

There are two solution concepts: weak and strong. We say that the process \( X \) is a **strong** solution if \( X(t) \) is \( \mathcal{F}_t \)-adapted, and if \( B(t) \) is given, it verifies the representation of the SDE.

An important special case is the **diffusion equation**, which is the SDE with constant coefficients and multiplicative noise

\[ dX(t) = aX(t)dt + \sigma X(t)dB(t). \]
As we already saw, the (strong) solution is the stochastic process $X$ such that

$$X(t) = xe^{(a - \frac{\sigma^2}{2})t + \sigma B(t)}$$

where $x$ is a random variable, which can be determined from $x = X(0)$, where $X(0)$ is a given initial distribution, and $B(t) = \int_0^t dB(s)$, if $B(0) = B_0 = 0$.

**Properties:**

1. asymptotic behavior:
   - if $a - \frac{\sigma^2}{2} < 0$ then $\lim_{t \to \infty} X(t) = 0$ a.s.
   - if $a - \frac{\sigma^2}{2} > 0$ then $\lim_{t \to \infty} X(t) = \infty$ a.s.
   - if $a - \frac{\sigma^2}{2} = 0$ then $\lim_{t \to \infty} X(t)$ will be finite a.s.

2. can we say anything about $E[X(t)]$?

$$E(X(t)) = E(X(0))e^{at}$$

To prove this, take Example 3 and observe that the stochastic integral of $Y(t) = e^{aB(t)}$ is

$$Y(t) = Y(0) + \frac{1}{2}a^2 \int_0^t e^{aB(s)} ds + a \int_0^t e^{aB(s)} dB(s)$$

taking expected values

$$E[Y(t)] = E[Y(0)] + E \left[ \frac{1}{2}a^2 \int_0^t e^{aB(s)} ds \right] + aE \left[ \int_0^t e^{aB(s)} dB(s) \right] =$$

$$= E[Y(0)] + \frac{1}{2}a^2 \int_0^t E \left[ e^{aB(s)} \right] ds + 0$$

because $e^{aB(s)}$ is a $f$-class function (from the properties of the Brownian motion. Differentiating

$$\frac{dE[Y(t)]}{dt} = \frac{1}{2}a^2 E[Y(t)]$$

as $E[Y(0)] = 1$. Then

$$E[Y(t)] = e^{\frac{1}{2}a^2t}$$

**References:** Oksendal (2003)
5.1.6 Stochastic optimal control

Finite horizon

We consider the stochastic optimal control problem, that consists in determining the value function, \( V(\cdot) \),

\[
V(t_0, x_0) = \max_u E_{t_0} \left[ \int_{t_0}^T f(t, x, u) dt \right]
\]

subject to

\[
dx(t) = g(t, x(t), u(t)) dt + \sigma(t, x(t), u(t)) dB(t)
\]

given the initial distribution for the state variable \( x(t_0, \omega) = x_0(\omega) \). We call \( u(\cdot) \) the control variable and assume that the objective, the drift and the volatility functions, \( f(\cdot) \), \( g(\cdot) \) and \( \sigma(\cdot) \), are of class \( H \) (the second) and \( N \) (the other two).

By applying the Bellman’s principle, the following nonlinear partial differential equation over the value function, called the Hamilton-Jacobi-Bellman equation, gives us the necessary conditions for optimality

\[
- \frac{\partial V(t, x)}{\partial t} = \max_u \left( f(t, x, u) + g(t, x, u) \frac{\partial V(t, x)}{\partial x} + \frac{1}{2} \sigma(t, x, u)^2 \frac{\partial^2 V(t, x)}{\partial x^2} \right).
\]

In order to prove it, heuristically, observe that a solution of the problem verifies

\[
V(t_0, x_0) = \max_u E_{t_0} \left( \int_{t_0}^T f(t, x, u) dt \right) = \max_u E_{t_0} \left( \int_{t_0}^{t_0+\Delta t} f(t, x, u) dt + \int_{t_0+\Delta t}^T f(t, x, u) dt \right)
\]

by the principle of the dynamic programming and the law of iterated expectations we have

\[
V(t_0, x_0) = \max_{u, t_0 \leq t_0 + \Delta t} E_{t_0} \left[ \int_{t_0}^{t_0+\Delta t} f(t, x, u) dt + \max_{u, t_0 \leq t_0 + \Delta t} E_{t_0+\Delta t} \left[ \int_{t_0+\Delta t}^T f(t, x, u) dt \right] \right] = \max_{u, t_0 \leq t_0 + \Delta t} E_{t_0} \left[ f(t, x, u) \Delta t + V(t_0 + \Delta t, x_0 + \Delta x) \right]
\]
if we write \( x(t_0 + \Delta t) = x_0 + \Delta x \). If \( V \) is continuously differentiable of the second order, the Itô’s lemma may be applied to get, for any \( t \)

\[
V(t + \Delta t, x + \Delta x) = V(t, x) + V_t(t, x)\Delta t + V_x(t, x)\Delta x + \frac{1}{2} V_{xx}(t, x)(\Delta x)^2 + h.o.t
\]

where

\[
\begin{align*}
\Delta x &= g\Delta t + \sigma \Delta B \\
(\Delta x)^2 &= g^2(\Delta t)^2 + 2g\sigma(\Delta t)(\Delta B) + \sigma^2(\Delta B)^2 = \sigma^2 \Delta t.
\end{align*}
\]

Then,

\[
V = \max_u E \left[ f\Delta t + V + V_t\Delta t + V_x g\Delta t + V_x \sigma \Delta B + \frac{1}{2} \sigma^2 V_{xx} \Delta t \right]
\]

as \( E_0(dB) = 0 \). Taking the limit \( \Delta \to 0 \), we get the HJB equation.

**Infinite horizon**

The autonomous discounted infinite horizon problem is

\[
V(x_0) = \max_u E_0 \left[ \int_0^\infty f(x, u)e^{-\rho t} dt \right]
\]

subject to

\[
dx(t) = g(x(t), u(t))dt + \sigma(x(t), u(t))dB(t)
\]

given the initial distribution of the state variable \( x(0, \omega) = x_0(\omega) \), and assuming the same properties for functions \( f(,), g(,) \) and \( \sigma(,) \). Also \( \rho > 0 \).

Applying, again, the Bellman’s principle, now the HJB equation is the nonlinear ordinary differential equation of the form

\[
\rho V(x) = \max_u \left( f(x, u) + g(t, x, u)V'(x) + \frac{1}{2} \sigma(x, u)^2 V''(x) \right).
\]

5.2 Applications

5.2.1 The representative agent problem

Here we present essentially the Merton (1971) model, which is a micro model for the simultaneous determination of the strategies of consumption and portfolio investment. We next present a simplified version with one risky and one riskless asset.

Let the exogenous processes be given to the representative consumer

\[ d\beta(t) = r\beta(t)dt \]
\[ dS(t) = \mu S(t)dt + \sigma S(t)dB(t) \]

where \( \beta \) and \( S \) are respectively the prices of the risky and the riskless assets, \( r \) is the interest rate, \( \mu \) and \( \sigma \) are the constant rates of return and volatility for the equity.

The stock of financial wealth is denoted by \( A(t) = \theta_0(t)\beta(t) + \theta_1(t)S(t) \), for any \( t \in T \).

Assume that \( A(0) = \theta_0(0)\beta(0) + \theta_1(0)S(0) \) is known.

Assume that the agent also gets an endowment \( \{y(t), t \in \mathbb{R}\} \) which adds to the incomes from financial investments and that the consumer uses the proceeds for consumption. Then the value of financial wealth at time \( t \) is

\[ A(t) = A(0) + \int_0^t (r\theta_0(s)\beta(s) + \mu \theta_1(s)S(s) + y(s) - c(s)) ds + \int_0^t \sigma \mu \theta_1(s)S(s)dB(s). \]

If the weight of the equity in total wealth is denoted by \( w = \frac{\theta_1 S}{A} \) then \( 1 - w = \frac{\theta_0 \beta}{A} \). Then, we get the differential representation of the instantaneous budget constraint comes

\[ dA(t) = [r(1 - w(t))A(t) + \mu w(t)A(t) + y(t) - c(t)]dt + w(t)\sigma A(t)dB(t). \tag{5.1} \]

The problem for the consumer-investor is

\[ \max_{c,w} E_0 \left[ \int_0^\infty u(c(t))e^{-\rho t}dt \right] \tag{5.2} \]
subject to the instantaneous budget constraint (5.1), given \( A(0) \) and assuming that the utility function is increasing and concave.

This is a stochastic optimal control problem with infinite horizon, and has two control variables. The Hamilton-Jacobi-Bellman equation is

\[
\rho V(A) = \max_{c,w} \left\{ u(c) + V'(A)[(r(1-w) + \mu w)A + y - c] + \frac{1}{2} w^2 \sigma^2 A^2 V''(A) \right\}.
\]

The first order necessary conditions allows us to get the optimal controls, i.e. the optimal policies for consumption and portfolio composition

\[
\begin{align*}
    u'(c^*) &= V'(A), \quad (5.3) \\
    w^* &= \frac{(r - \mu)V'(A)}{\sigma^2 AV''(A)} \quad (5.4)
\end{align*}
\]

If \( u''(.) < 0 \) then the optimal policy function for consumption may be written as \( c^* = h(V'(A)) \). Plugging into the HJB equation, we get the differential equation over \( V(A) \)

\[
\rho V(A) = u \left( h(V'(A)) \right) - h(V'(A))V'(A)(y + rA)V'(A) - \frac{(r - \mu)^2(V'(A))^2}{2\sigma^2 V''(A)}. \quad (5.5)
\]

In some cases the equation may be solve explicitly. In particular, let the utility function be CRRA as

\[
u(c) = \frac{c^{1-\eta} - 1}{1 - \eta}, \quad \eta > 0
\]

and conjecture that the solution for equation (5.5) is of the type

\[
V(A) = x(y + rA)^{1-\eta}
\]

for \( x \) an unknow constant. If it is indeed a solution, there should be a constant, dependent upon the parameters of the model, such that equation (5.5) holds.

First note that

\[
\begin{align*}
    V'(A) &= (1 - \eta)rx(y + rA)^{-\eta} \\
    V''(A) &= -\eta(1 - \eta)r^2x(y + rA)^{-\eta-1}
\end{align*}
\]
then: the optimal consumption policy is

$$c^* = (xr(1 - \eta))^{-\frac{1}{\eta}} (y + rA)$$

and the optimal portfolio composition is

$$w^* = \left( \frac{\mu - r}{\sigma^2} \right) \frac{y + rA}{\eta r A}$$

Interestingly it is a linear function of the ratio of total (human plus financial wealth \(\frac{y}{r} + a\)) over financial wealth.

After some algebra, we get

$$V(A) = \Theta \left( \frac{y + rA}{r} \right)^{1-\eta}$$

where

$$\Theta \equiv \frac{1}{1 - \eta} \left[ \frac{\rho r (1 - \eta)}{\eta} - \frac{(1 - \eta)}{2\eta^2} \left( \frac{\mu - r}{\sigma} \right)^2 \right]^{-\eta}$$

Then the optimal consumption is

$$c^* = \left( \frac{\rho r (1 - \eta)}{\eta} - \frac{(1 - \eta)}{2\eta^2} \left( \frac{\mu - r}{\sigma} \right)^2 \right) \left( \frac{y + rA}{r} \right)$$

If we set the total wealth as \(W = \frac{y}{r} + A\), we may write the value function and the policy functions for consumption and portfolio investment

$$V(W) = \Theta W^{1-\eta}$$
$$c^*(W) = (1 - \eta) \Theta^{-\frac{1}{\eta}} W$$
$$w^*(W) = \left( \frac{\mu - r}{\eta \sigma^2} \right) \frac{W}{A}$$

**Remark** The value function follows a stochastic process which is a monotonous function for wealth. The optimal strategy for consumption follows a stochastic process which is a

\(^1\)Of course, \(x = r^{-(1-\eta)} \Theta\).
linear function of the process for wealth and the fraction of the risky asset in the optimal portfolio is a direct function of the premium of the risky asset relative to the riskless asset and is a inverse function of the volatility.

We see that the consumer cannot eliminate risk, in general. If we write $c^* = \chi A$, where $\chi \equiv (1 - \eta)\Theta^{-\frac{\mu}{\sigma^2/2}}$, then the optimal process for wealth is

$$dA(t) = [r^* + (\mu - r)w^* - \chi]A(t)dt + \sigma w^* A(t)dB(t)$$

where $r^* = r\frac{W}{A}$, which is a linear SDE. Then as $c^* = c(A)$, if we apply the Itô’s lemma we get

$$dc = \chi dA = c(\mu_c dt + \sigma_c dB(t))$$

where

$$\mu_c = \frac{r - \rho}{\eta} + \frac{1 + \eta}{2} \left( \frac{\mu - r}{\sigma \eta} \right)^2$$

$$\sigma_c = \frac{\mu - r}{\sigma \eta}.$$ 

The sde has the solution

$$c(t) = c(0) \exp \left\{ \left( \mu_c - \frac{\sigma_c^2}{2} \right) t + \sigma_c B(t) \right\}$$

, where $\frac{\mu - r}{\sigma}$ is the Sharpe index, and the unconditional expected value for consumption at time $t$

$$E_0[C(t)] = E_0[C(0)]e^{\mu_c t}.$$ 


5.2.2 The stochastic Ramsey model

Let us assume that the economy is represented by the equations

$$dK(t) = (F(K(t), L(t)) - C(t))dt$$

$$dL(t) = \mu L(t)dt + \sigma LdB(t)$$
where we assume that $F(K, L)$ is linearly homogeneous, given the (deterministic) initial stock of capital and labor $K(0) = K_0$ and $L(0) = L_0$.

The growth of the labor input (or its productivity) is stochastic.

Let us define the variables in intensity terms

$$k(t) \equiv \frac{K(t)}{L(t)}, \quad c(t) \equiv \frac{C(t)}{L(t)}$$

We can get the restriction of the economy as a single equation on $k$ by using the Itô’s lemma

$$dk = \frac{\partial k}{\partial K}dK + \frac{\partial k}{\partial L}dL + \frac{1}{2} \frac{\partial^2 k}{\partial K^2}(dK)^2 + \frac{1}{2} \frac{\partial^2 k}{\partial K\partial L}dKdL + \frac{1}{2} \frac{\partial^2 k}{\partial L^2}(dL)^2$$

$$= \frac{F(K, L) - C}{L}dt - \frac{K(\mu dt + \sigma dB(t))}{L^2}dt + \sigma^2 \frac{K}{L}dt$$

if we set $f(k) = F \left( \frac{K}{L}, 1 \right)$.

The HJB equation is

$$\rho V(k) = \max_c \left\{ u(c) + V'(k) \left( f(k) - c - (\mu - \sigma^2)k \right) + \frac{1}{2} (k\sigma)^2 V''(k) \right\}$$

the optimality condition is again

$$u'(c) = V'(k)$$

and, we get again a 2nd order ODE

$$\rho V(k) = u(h(k)) + V'(k) \left( f(k) - h(k) - (\mu - \sigma^2)k \right) + \frac{1}{2} (k\sigma)^2 V''(k).$$

Again, we assume the benchmark particular case: $u(c) = \frac{c^{1-\theta}}{1-\theta}$ and $f(k) = k^\alpha$. Then the optimal policy function becomes

$$c^* = V'(k)^{-\frac{1}{\theta}}$$

and the HJB becomes

$$\rho V(k) = \frac{\theta}{1-\theta} V'(k)^{\frac{\theta-1}{\theta}} + V'(k) \left( k^\alpha - -(\mu - \sigma^2)k \right) + \frac{1}{2} (k\sigma)^2 V''(k).$$
We can get, again, a closed form solution if we assume further that $\theta = \alpha$. Again we conjecture that the solution is of the form

$$V(k) = B_0 + B_1 k^\alpha$$

Using the same methods as before we get

$$B_0 = (1 - \alpha) \frac{B_1}{\rho} \quad B_1 = \frac{1}{1 - \alpha} \left[ \frac{(1 - \alpha) \theta}{(1 - \theta)(\rho - (1 - \alpha)^2 \sigma^2)} \right]^\alpha.$$

Then

$$V(k) = B_1 \left( \frac{1 - \alpha}{\rho} + k^{1 - \alpha} \right)$$

and

$$c^* = c(k) = \left( \frac{(1 - \theta)(\rho - (1 - \alpha)^2 \sigma^2)}{(1 - \alpha) \theta} \right) k \equiv \varrho k$$

as we see an increase in volatility decreases consumption for every level of the capital stock.

Then the optimal dynamics of the per capita capital stock is the SDE

$$dk^*(t) = \left( f(k^*(t)) - (\mu + \varrho - \sigma^2)k^*(t) \right) dt - \sigma^2 k^*(t) dB(t).$$

In this case we can not solve it explicitly as in the deterministic case.

References: Brock and Mirman (1972), Merton (1975), Merton (1990)
Bibliography


