Spatial heterogeneity and growth

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Abstract

In this paper we deal with the interaction between capital accumulation through time and migration across space. We extend in the simplest way the two benchmark models in the economic growth literature: Solow's and Ramsey's models. In the first case, space is introduced merely through a balance equation and in the second by considering aggregation among heterogeneous consumers through a Benthamian preference criterium. In both cases, the space-time dynamics leads to parabolic partial differential equations. We study qualitatively both models by assuming that solutions are of the traveling wave type. We found that, in the capital accumulation model, if traveling waves exist, their speed of propagation through time is infinite. In the optimal growth and distribution model traveling waves do not exist.

KEYWORDS: Spatial growth; Optimal growth and distribution. Partial differential equations; Optimal control of partial differential equations; Traveling waves.

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1 Introduction

One of the major recent research topics in both macroeconomic and growth theories deals with the existence of some form of heterogeneity. Theory is shifting from concentrating only on the behavior of aggregates through time towards addressing and incorporating evolution of distributions.

This is related, in part, with the still poor tool kit in which the economic profession is trained. An obvious modeling framework for the evolution of distributions are partial differential equations (PDE), and, in particular, of the parabolic kind.

In other scientific areas, as physics, biology, geography and demography some phenomena involving space-time dynamics are represented by PDE. For instance the dynamics of heath, waves, epidemics, populations, etc.

However PDE's have not been completely absent to economics. In finance, their use started in the early 70's in models for pricing contingent claims in complete markets with risk neutrality, following [3] and [15]. Basically, all option pricing models assume that the price of the underlying asset follows a diffusion process, or that the information revelation mechanism is a filtration over a Brownian motion, and its pricing equation is a parabolic PDE (or a partial differential inequality) with more or less complicated boundary conditions (classic or free) [1]

In economic theory, the representation of models by PDE has arisen in several contexts: in the target zones models, which are basically open economy ISLM models with uncertainty (see [11]); on the spatial dynamics, and, in particular on the dynamics of agglomeration (see [9] for a first contribution and for more recent approaches see [2], or [20], chap. 3.); on the spatial dynamics of prices (see [28]); on migration see [16]; or in vintage models of the capital stock (see [24]).

It seems surprising that this formal structure has had such a small impact on the recent macroeconomic literature, given its potential for modeling in the presence of heterogeneous agents, incomplete markets in general equilibrium and the redistributive consequences of fiscal policy. [2]

In this paper, we deal with growth theory within a explicitly modeled spatial heterogeneous economy. As with the macro literature, central distributional aspects of growth, as convergence between growing countries or regions have been dealt in the literature in a a-spatial context [3]. The basic

[1] [10], chap. 4 explicitly deals with the relationships between the PDE and the Brownian motion.

[2] See [23] for a recent survey of this literature.

[3] See for instance [5] and [19].

simplifying assumptions usually made tend to imply either the elimination of the dynamics (by dealing only with the steady state distribution) or the spatial distribution (by reducing all the analysis to the comparative dynamics of only two points in the support. In particular, the convergence controversy in growth theory between Sala-y-Martin and Quah, on the $\beta-$ and $\sigma-$, convergences may be clarified by using a parabolic PDE [4].

We extend the capital accumulation model by Solow [26] in the simplest way, by introducing space within a regional budget constraint equation. Assuming that an economy is composed by spatially heterogeneous regions, if these regions are not isolated then capital will flow not only across time but also across space. If a region has positive savings it will invest in own financial assets and in assets issues by neighboring regions. While the first investment will raise its productive capacity, the latter will increase the productive capacity of neighboring regions.

We study the qualitative properties of the solution, by assuming that they are of the traveling wave type. That is, that there is a wave that evolves through time at a given speed while keeping its basic form. We found that for the existence of traveling waves they should have infinite speed. This result is not surprising, because we did not introduce any dampening effect in the geographical distribution of capital. The consequence of this is that the Solow model is compatible with the presence of domestic capital flows in a perfect capital market economy.

The second model is a extension of the optimal growth model with cross-sectional heterogeneity. Therefore, we address the existence and properties of optimal distributive policies. Now, differently from the aggregate intertemporal optimization, the inter-personal aggregation optimization is not so undisputable. We may have several optimality criteria (Bentham-Ian, Rawlsian, Von-Neumann-Morgenstern, etc) that may lead to different optimal distributional policies. In particular, we assumed a Benthamian benevolent dictator which allocates consumption both through time and space in order to maximize the present value an un-weighted aggregate utility function. Again, we studied the optimal PDE system for the presence of traveling waves. In this case traveling waves will not exist.

Next, in section 2 we present and study a model with capital accumulation and migration and in section 3 the optimal intertemporal distribution model. The last section concludes.

[4]See [25] and [21]. [13] is a canonic quotation in this literature, as a pathbreaking paper on the theory of income distribution. We think that the ensuing analysis is both simpler and more powerful.

2 Capital accumulation and migration

In this section we will consider the spatial dimension in the framework of the Solow [26] model.

Let us assume that there is heterogeneity among economic units spread out across the continuum $X = (-\infty, +\infty)$. Call it space. At a given moment t the unit located at x is characterized by the stock of capital and the level of consumption, $k(t, x)$ and $c(t, x)$, respectively, where $(t, x) \in (\mathbb{T}, X) \subset \mathbb{R}^+ \times \mathbb{R}$. In particular, the initial distribution of the capital stock endowments is given by $k(0, w) = k_0(w)$, and we may assume Cauchy boundary conditions as $\lim_{x\to\pm\infty} k(x, t) = 0$. With the capital stock allocated to productive activities, each region produces $y(t, x) = f[k(t, x)]$. We assume that the technology is the same across different regions and that the differences in the marginal productivity of capital are only related to the different capital endowments. The technology is neoclassical: we need some input to produce a positive quantities ($f(0) = 0$, $f(.) > 0$ for $k > 0$), the marginal returns are positive ($f'(.) > 0$ and Inada), but are decreasing ($f^{''} < 0$). To simplify assume a Cobb-Douglas production function:

$$y = Ak^{\alpha}, \quad 0 < \alpha < 1$$

where A is an exogenous constant and homogeneous Hicks-neutral technology parameter.

Therefore

$$s(t, x) = f[k(t, x)] - c(t, x)$$

are the savings for each location at time t. . In order to stay close as possible to Solow's [26] model, assume that $c(t, x) = (\beta - 1) f[k(t, x)]$, with $\beta \in (0, 1)$. Then as $s(t, x) = \beta f[k(t, x)]$, beta is the marginal propension to saving at the location x.

Therefore, the only source of heterogeneity is related to the differences in the spatial allocation of the stock of capital.

All the previous variables are defined in per capita terms. We are implicitly assuming that population distribution may change through time such that $\int_X \eta(t, x)dx = 1$ identically for all t. As there is no population migration we may assume that it is evenly distributed across space and exclude it from the subsequent analysis.

A region is an aggregation of neighboring locations, that it $R \subseteq X$. For region R, the corresponding aggregate are $K(t) = \int_R k(t, x)dx$, $S(t) = \int_R s(t, x)dx$, for the capital stock and flow of savings, for a particular moment. If we assume that the capital stock depreciates a constant homogeneous rate $\rho \in (0,1)$, then the aggregate net investment is

$$\frac{dK}{dt} - \rho K(t) = \int_R \frac{\partial k(t,x)}{\partial t} dx - \rho \int_R k(t,x) dx = \int_R \left[\frac{\partial k(t,x)}{\partial t} - \rho k(t,x) \right] dx.$$

In this setting, the regional equilibrium changes both across time and space. That is, there is both capital accumulation within each region and capital migration. As we assume that there are no costs of adjustment in investment and that the capital market is perfect, then k represents both physical capital and equity, i.e., financial claims to capital.

Let us separate the two types of dynamics, by dealing first with capital accumulation through time and introducing next the capital migration across space.

2.1 Autarky

Assume that there are no capital flows between regions. As production, consumption, investment and financing activities are carried on in autarky, then net investment is permanently equal to consumption, in each region. Then, by aggregation, we will have the macroeconomic equilibrium, $\frac{dK}{dt} - \rho K(t) = S(t)$,

$$\int_R \frac{\partial k(t,x)}{\partial t} - \rho k(t,x) - s(t,x) dx = 0 \quad \forall t, \tag{1}$$

or, equivalently $\frac{\partial k}{\partial t} = s - \rho k$. If capital flows internally to each region, there will be spatial homogeneity intra-walls then, $k(t,x) = k(t)$ and we get a version of the Solow capital accumulation equation

$$\frac{dk(t)}{dt} = \beta f(k(t)) - \rho k(t),$$

meaning that when there is excess savings over capital depreciation, capital will accumulate through time in each region.

Then this is a model for the so-called β-convergency: if $\beta \frac{\partial f}{\partial k} < \rho$, for any x, then each country will tend to its own steady state equilibrium. Convergence is only constrained by the fact that, for each country, the initial endowments may be different from their own steady state stock of capital. The marginal productivity of capital may be and will keep being different across countries along the transition path. As we assume that all the parameters are equal across regions, they will tend to the same steady-state.

If we assumed that there is spatial heterogeneity in the behavioral primitive parameters tend each region will tend to its own different steady state capital stock.

Though we may determine an evolution of the capital distribution along time it will be independent of any spatial dynamics. Without any further structure in the model, in particular without any further assumption over the spatial distribution of the marginal productivity and of the saving rates, no conclusions may be drawn as regards σ-convergence.

2.2 Capital migration

If capital is allowed to flow between different regions, then the equilibrium condition (1) will not hold. Now, capital will not only accumulate through time, but it will also flow to different regions. Here, we will assume the simplest setup in which the flow of capital will residually flow as a consequence of production and savings decisions. There are also no imperfections in the inter-regional capital markets that will impose any extra premium or discount on capital movements.

In the open region, the microeconomic equilibrium condition is such that the aggregate savings is equal to aggregate gross investment plus net capital outflow, or in other words, the financing capacity is invested in domestic assets and in foreign assets,

$$\int_R s(t,x)dx = \int_R \left(\frac{\partial k(t,x)}{\partial x} + \rho k(t,x) \right) dx + \int_{\partial R} -\frac{\partial k}{\partial x} dn = 0, \qquad \forall t.$$

The net capital outflow, in the present set up, may be measured as the aggregate flux of capital $(\frac{\partial k}{\partial x})$ that passes through the boundaries of the region (∂R) in the outward direction (minus sign) instantaneously.

As each region has a boundary, δR, composed of two borders, capital may flow in different directions in each border. Therefore, the net capital outflow is given formally by the line integral $\int_{\partial R} \mu dn$, where n is the vector which is normal to the boundary ∂R. From the divergence theorem, we may determine the net capital flow through the region as the common integral,

$$\int_{\partial R} \mu dn = \int_R \text{div}(\mu) dx = \int_R \frac{\partial \mu}{\partial x} dx,$$

where div is the divergence operator [5]. Then the following macroeconomic

[5] See, for those concepts in advanced calculus [14].

equilibrium for region R results,

$$\int_R \left[\frac{\partial k(t,x)}{\partial t} - \rho k(t,x) - \frac{\partial^2 k}{\partial x^2} - s(t,x) \right] dx = 0 \quad \forall t.$$

As this equilibrium has to hold for each and every region, then we get the following quasi-linear parabolic partial differential equation

$$\frac{\partial k}{\partial t} = \frac{\partial^2 k}{\partial x^2} + f(k) - c - \rho k,$$

which gives us the behavioral equation for the capital stock in a given space-time point.

The spatial extension of the Solow accumulation equation is

$$\frac{\partial k}{\partial t} = \frac{\partial^2 k}{\partial x^2} + \beta A k^{\alpha} - \rho k = \Upsilon(t,x). \tag{2}$$

Let the associated traveling wave be written as s as $k(t,x) = w(\xi)$ where $\xi = x - ct$ and c is the unknown speed of propagation of the wave. Then, (2) may be equivalently written as a second order ordinary differential equation

$$\ddot{w} + c\,\dot{w} + \beta A k^{\alpha} - \rho w = 0, \tag{3}$$

or has a system of first-order ODE's

$$\begin{aligned} \dot{w} &= p \\ \dot{p} &= -cp - \beta A k^{\alpha} + \rho w. \end{aligned} \tag{4}$$

For a scalar equation the only type of monotonous traveling waves that may exist are front waves. A front wave is a solution of equations (3) or (4) such that

$$\lim_{\xi \to \pm\infty} w(\xi) = w_{\pm}, \qquad w_{+} \neq w_{-}.$$

If w is a bounded function then the stationary points and their local stability properties may be determined from the zeros of the kinetic non-distributed equation

$$\dot{k}(t) = \beta A k^{\alpha} - \rho k.$$

Therefore, we get

$$w_{+} = 0 \quad w_{-} = \left(\frac{\beta \alpha A}{\rho} \right)^{\frac{1}{1-\alpha}},$$

and w_+ is locally unstable and w_- is locally stable because

$$\frac{\partial F(w_+)}{\partial w} = +\infty, \qquad \frac{\partial F(w_-)}{\partial w} = (\alpha - 1)\rho < 0.$$

Equivalently, the equilibrium points of the (4) system are $(0, w_+) = (0, 0)$ and $(0, w_-)$. The first is a saddle point (with $\pm\infty$ eigenvalues and the second is a stable node.

Volpert et al [27] call this the monostable case. Then according to the existence theorem in [27] the parabolic PDE admits a stationary traveling wave solution, of the monotonous wave front type, if there is a critical value for the wave speed of propagation (c^*) such that if $c \geq c^*$. This wave is stationary in the sense that it travels across space with speed c keeping the same shape through time.

The critical value for the speed of propagation that allows for the existence of a wave front may also be computed (see [27], page 22): as $F(.) := \beta A\mu^\alpha - \rho\mu > 0$ for $w_+ < \mu < w_-$ and $F'(w_+) \geq F'(\mu)$ for for $w_+ \leq \mu \leq w_-$, then $c^* = 2\left(F'(w_+)\right)^{\frac{1}{2}} = +\infty$.

Therefore, if it exists, the stationary wave fronts should propagate at an infinite speed. If we consider the space-time frame for $\xi = x - ct$, the critical speed corresponds to a case in which the space dimension is not important for the dynamics of capital stock.

The intuition for this may be the following. If there is an initial heterogenous distribution of capital $k_0(x)$, the regions with the lowest capital stock will have higher marginal productivities of capital. If capital travels in search for higher real interest rates, then it will flow from the locations with higher initial endowments of capital to the locations with the lowest. This will imply a tendency for the equalization of the capital stock and, therefore, of the interest rates across regions. As there are no barriers institutional or not to regional capital flows, the capital will flow infinitely fast. This is what we should expect with perfect capital markets [6].

2.3 Convergence

In this model, in addition to β-convergence we may also discuss σ-convergence. The last type of convergence is related to change in the inter-spatial dispersion in the beginning and in the steady state. We get this by using equation (2) and making $\frac{\partial k}{\partial t} = 0$. In the beginning if $k(0, x) = k_0(x)$ and if we take the variance as measure of dispersion, then $\sigma_0 := \int_x x^2 k_0(x)dx$. In the steady

[6] [2] has similar results as regards the spatial diffusion of prices and innovation.

state , if $\Upsilon(.)$ is stationary we will have $k_\infty(x) := \{k(t,x) : \lim_{t\to\infty} \Upsilon(t,x) = 0\}$ and the terminal variance is measured by $\sigma_\infty := \int_x x^2 k_\infty(x)dx$. As capital will collapse towards $k_- = w_-$, then there complete $\sigma-$convergence as $\sigma_\infty = 0$.

Though the asymptotic result does not seem to be unacceptable in the very long run, for our present time scale it is counterfactual. We tend to observe, for instance in the distribution of income across countries (see Quah [21]) a bimodal distribution. We do not know if that distribution has the same form as the asymptotic distribution or if it is only a transitional distribution. In any case, the previous simple extension of the Solow model cannot account for this type of transitional or asymptotic distributions.

There are other different forms of regional heterogeneity that we may assume, within the same kind of framework, other sources of heterogeneity: in the exogenous productivity factor $A = A(t,x)$, in the share of capital $\alpha = \alpha(t,x)$ or in savings behavior $\beta = \beta(t,x)$.

We incorporate the last kind of heterogeneity in a centralized economy model.

3 Optimal distribution and growth

Assume that there is a benevolent dictator that allocates consumption across regions in order to maximize an intertemporal aggregate welfare function.

The Ramsey model (see [22]) gives a benchmark for most of recent economic growth and macroeconomic theories, both in centralized and in decentralized economies. That model addresses the problem of the choice of the optimal trajectory of consumption when there is a trade-off between present and future consumption. However, it assumes a representative agent economy without heterogenous agents. Equivalently it may be thought as a model for a representative agent or region in an athomistic economy.

If we assume an economy with heterogeneous agents, while keeping the centralized setting, the central planner should choose not only the optimal intertemporal allocation of consumption, but also the optimal intratemporal distribution. There are two fundamental spatial dimensions which we do not find in the former model: first, optimal redistributive policies change the distribution of capital among different regions, and, second, a criteria for aggregating utilities across space should be made explicit.

While there is a largely undisputable criterium for aggregating utilities intertemporally (maximization of the actual value of present and future utilities), there several possible collective preference functions [7]. Several interpersonal or interspatial preference criteria may be found in the literature (see [1] for the static counterpart) Benthamian, Von-Neumann-Morgenstern, egalitarian and Rawlsian. The first two, are average weighted aggregate utility functions. However, while Benthamian utilities assume an equal weight, vNM functions average utility functions. Egalitarian aggregate utility functions may be thought as minimizing the square of the deviations from the personal utility levels from a unique common benchmark and Rawlsian utility function may be seen as minimizing the variance of the aggregate utility function.

Next we will assume a Benthamian utility criterium, assuming that the budget constraint for the planer is given by the accumulation function studied in the previous section.

3.1 A Benthamian planner

The Benthamian objective function is an aggregate utility function which involves a simple, un-weighted, sum of the individual utility functions. In the present intertemporal (dynamic as opposed to the static case as in [1]) assume that the consumers are distributed in the continuum, and assume that in each region there is a representative consumer, and that the utility functions are "spatially" homogeneous. The centralized problem is to maximize

$$\int_X \int_0^\infty u(c(t,x))e^{-\delta t} dt dx,$$

subject to

$$\begin{cases} \frac{\partial k}{\partial t} = \frac{\partial^2 k}{\partial x^2} + Ak^\alpha - c - \rho k \\ k(0,x) = k_0(x). \end{cases}$$

The Hamiltonian is

$$H = u(c)e^{-\delta t} + p(Ak^\alpha - c - \rho k),$$

and the optimal consumption policy is

$$\frac{\partial u[c^*(t,x)]}{\partial c} = p(t,x)e^{\delta t}$$

[7] We expressly avoid entering the vast field of collective preference functions, and aggregation criteria, in the sequel of [8].

where the co-state variable is determined from

$$\begin{cases} \frac{\partial p}{\partial t} + \frac{\partial^2 p}{\partial x^2} = -p[A\alpha k^{\alpha-1} - \rho) & \forall (x,t) \in - \\ \lim_{x\to\pm\infty} p(x,t) = 0 & \forall t \\ \lim_{t\to\infty} \frac{\partial p}{\partial t} = 0 & \forall x \end{cases}$$

If we define the co-state variable in current terms as $q(t,x) = e^{\delta t} p(t,x)$ and, if, the utility function is logarithmic, then

$$c^*(t,x) = \frac{1}{q(t,x)} \tag{5}$$

and

$$\begin{cases} \frac{\partial q}{\partial t} + \frac{\partial^2 q}{\partial x^2} = q(\delta + \rho - A\alpha k^{\alpha-1}) & \forall (x,t) \in - \\ \lim_{x\to\pm\infty} q(x,t) = 0 & \forall t \\ \lim_{t\to\infty} \frac{\partial q}{\partial x} e^{-\delta t} = 0 & \forall x. \end{cases} \tag{6}$$

Then we will have a Cauchy well-posed problem which may be solved backwards from $t = \infty$, for the dual price. From equation (6) we also get a trajectory for the distribution of consumption across time.

Several comparisons may be made as regards the original version of the Ramsey model: First, the static determination of the optimal consumption policy is formally identical. Second, there are intertemporal spatial reallocations of consumption, which are induced by the diffusion term in the dynamic equation for the dual price of consumption, as it may be seen in PDE (6). In other words, there are two dimensions for the spatial diffusion: the first is related to the tendency for the spatial equalization of the marginal productivity of capital and the second is related with the spatial evaluation of consumption by the central authority. At last, it can be seen that, even in the steady state, the net investment, which should be equal to the capital depreciation, should be equal to savings plus the income resulting from inter-spatial transfers.

3.2 Traveling wave solutions

Next we will address the existence of traveling waves. Let

$$q(t,x) = w_1(x - ct) \quad k(t,x) = w_2(x - ct)$$

where c is the speed of propagation. Then, traveling waves, if they exist, should be the solution of the system

$$\ddot{w}_1 - c\,\dot{w}_1 - w_1\left(\delta + \rho - \alpha A w_2^{\alpha-1}\right) = 0 \tag{7}$$

$$\ddot{w}_2 - c\,\dot{w}_2 - \left(A w_2^{\alpha} - \rho w_2 - \frac{1}{w_1}\right) = 0 \tag{8}$$

The equilibrium points for the associated non-distribution kinetic system are determined, in the space (w_1, w_2), by the intersection of the two isoclines $\frac{dw_1}{dt} = \frac{dw_2}{dt} = 0$. The first isocline is given by the curves $w_1 = 0$ or $w_2 = \left(\frac{\alpha A}{\delta+\rho}\right)^{\frac{1}{1-\alpha}}$ and the second by $1 = w_1\left(Aw_2^{\alpha} - \rho w_2\right)$. It is easy to see that there is only one bounded equilibrium point [8]. Therefore there are no traveling wave solutions for this model.

The intuition for this may be the following: if there is any local change in consumption and therefore in savings induced by the central authority, as capital flows perfectly across regions, therefore the redistributive policy is completely annihilated in the transition path. Therefore, only the intertemporal dimension matters, and there is not a stationary space-time pattern between the capital stock and consumption "waves".

This is in part a consequence of the fact that the central authority has not a particular preference for consumption in certain regions.

4 Concluding remarks

In this paper we wanted to show the potential that PDE's and the optimal control of PDE offer for modeling heterogeneity in economic models.

We extended in the simplest and more neutral way two benchmark models in economic theory: the Solow [26] and the Ramsey [22] models. We may say that we got a confirmation that in the simplest setups the spatial dimension is unimportant. *A contrario*, we may interpret those models as space-time models in which inter-regional capital markets are perfect and the central planner has a Benthamian preference criterium.

[8]The second curve has the asymptote $w_1 = 0$. So, we may roughly say that there is another unbounded equilibrium point at $(+\infty, 0)$.

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A Traveling wave solutions for parabolic PDE systems

Next, we follow the presentations by [7] and [27]). Consider the Cauchy problem

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + F(u) & (x,t) \in - \\ u(x,0) = f(x) & \forall x \end{cases}$$

where $u : - \rightarrow \mathbb{R}^n$, $- = (-\infty, +\infty) \times [0, +\infty)$. Assume that $F(u)$, the source, is continuously differentiable and $f(x)$ is piecewise continuous and bounded. A *traveling wave* solution is a solution of as function of the form

$$u(x - ct) = w(\xi)$$

such that w is twice-continuously differentiable and bounded on $(-\infty, +\infty)$. c is the unknown speed of the wave. Therefore, function $w(\xi)$ satisfies the second order system of ODE equations

$$\begin{cases} \ddot{w} + c\,\dot{w} + F(w) = 0 & \forall \xi \in (-\infty, +\infty) \\ \lim_{\xi \rightarrow \pm\infty} = w_{\pm} \end{cases} \tag{9}$$

where $\dot{w} = \frac{\partial w}{\partial \xi}$, or can be reduced to the equivalent first order ODE system

$$\begin{cases} \dot{w} = p \\ \dot{p} = -cp - F(w), \\ \lim_{\xi \rightarrow \pm\infty} = w_{\pm} \end{cases} \tag{10}$$

We call waves, the solutions of systems (9) or (10).

There are several types of waves. For the one-dimensional state spaces there only two types: stationary waves which may be monotone (wave fronts), or non-monotone (pulse waves and periodic waves) or systems of waves (wave trains) and non-stationary waves.

The existence, characterization and the stability properties of waves depend upon the properties of the source function $F(w)$.

For the existence of waves, the two limits for w should exist such that

$$w_+ \leq w(\xi) \leq w_- \qquad -\infty < \xi + \infty$$

holds, the derivatives of w tend to zero as $\xi \rightarrow \pm\infty$ and are the zeros of $F(w)$. Then a necessary condition for a wave to exist is that the function $F(.)$ should have at least two zeros.

Additionally, if a wave exists, the properties of the source allows us to classify it and study its stability. For one dimensional state space, the following types of waves may exist:

- if the equilibrium points are different, $w_+ \neq w_-$, then we may have a wave front if the eigenvalues of $F(.)$ are real and there is at least one negative eigenvalue, then wave fronts will exist if they are all negative or if the speed of propagation is larger than a critical value;
- if $w_+ = w_-$ and if there is an homoclinic loop then waves exist and take the form of pulses;
- if there is a limit cycle then periodic traveling waves will exist.

In the models studied in this paper we may have sources or saddles then wave fronts will exist if $c \geq 2(F(w_+))^{\frac{1}{2}}$ if $n = 1$. (see [27] theorem..).

Stability in this context is related to the approach to a traveling wave.

B Optimal control of PDE's

Consider a state variable $z : - \;\rightarrow \mathbb{R}^n$, where $- \; = \mathbb{R}^+ \times \mathbb{R}$ mapping $(t, x) \mapsto z(t, x)$ and the quasi-linear PDE

$$\mathcal{L}z = f(z, u; t, x) \tag{11}$$

perturbed by a parameter $u \in \mathbb{R}^m$, where $\mathcal{L} := \alpha(t, x)\frac{\partial^2}{\partial x^2} + \beta_x(t, x)\frac{\partial}{\partial x} + \beta_t(t, x)\frac{\partial}{\partial t}$ is the parabolic operator. Now assume that u is a variable $u : - \;\rightarrow \mathbb{R}^m$ mapping $(t, x) \mapsto u(t, x)$ and that it is chosen optimally. The associated optimal control problem consists in determining endogenously the control vector such that the functional is maximized

$$\max \int F(u, z; t, x)dv + \int_{\partial} S(z; t, x)$$

in which $dv = dt.dx$ and S is a 1-form. We also have to impose boundary conditions, for instance Cauchy or Dirichlet boundaries. This problem is also called the optimal control problem for distributed parameter problems [9].

The application of the variational principle leads to the following extension of the Pontryiagin's maximum principle, for the necessary first order conditions: there is a function $p : - \;\rightarrow \mathbb{R}$, mapping $(t, x) \mapsto p(t, x)$, piecewise continuous such that, the stating the Hamiltonian as

$$H(u, z, p; t, x) = F(z, u; t, x) + pf(z, u; t, x),$$

[9] See [4], [12] [24], [18], [6] or [17] for expositions and demonstrations of the next results, with varying generality.

the optimal solutions for u and z, u^* and z^*, verify, for each (t, x) and for u admissible

$$u^* = \text{argmax}_u H(u, z, p; t, x),$$

where

$$\mathcal{L}^* p = -\frac{\partial H}{\partial z}$$

and $\mathcal{L}^* := -\alpha(t,x)\frac{\partial^2}{\partial x^2} + \beta_x(t,x)\frac{\partial}{\partial x} + \beta_t(t,x)\frac{\partial}{\partial t}$ is the dual parabolic operator.

The dual boundary conditions depend upon the type of the problem. For the Cauchy problem, the dual boundary conditions are

$$\begin{cases} p(\xi, t) = 0 & \text{if } \xi \in \partial X, \ \forall t \\ \lim_{t\to\infty} \frac{\partial p(t,x)}{\partial} = \frac{\partial S}{\partial z} & \forall x \in X \end{cases}$$

and

$$\begin{cases} \frac{\partial p(\xi,t)}{\partial n} = \frac{\partial S}{\partial z} & \text{if } \xi \in \partial X, \ \forall t \\ \lim_{t\to\infty} p(t,x) = \frac{\partial S}{\partial z} & \forall x \in X \end{cases}$$

for the Dirichlet problem, where n is the exterior normal to ∂- .