A spatial Solow model with unbounded growth

Paulo Brito
UECE-ISEG,
Universidade Técnica de Lisboa
Email: pbrito@iseg.utl.pt

31.12.2004

Abstract

In this paper we address unbounded growth in a spatially heterogeneous world. We present a continuous time-continuous space case extension of the Solow (1956) model, where capital and labor mobility generate two spatial interacting forces. The first endogenously generates a distributed balanced growth path and the second is assumed to be exogenous. Our approach, formalized by a non-linear parabolic partial differential equation, allows for an integrated treatment of β- and σ-convergence, which is studied in growth theory, and dispersion and agglomeration, studied in the new geographic economics. After proving the existence of stability, in a distributional sense, we consider two types of disturbances, with or without a dispersive effect, as a way of qualitatively studying the dynamic properties of the model. In the first case we get convergence towards a symmetric BGP and in the second we will have an exogenously generated agglomeration, converging towards an asymmetric BGP.

KEYWORDS: spatial growth, partial differential equations, Fourier transforms

JEL CLASSIFICATION: C6, D9, E1, R1.

[0]Please address correspondence to: Paulo Brito, ISEG, Rua Miguel Lupi, 20, 1249-078 Lisboa, Portugal. Email: pbrito@iseg.utl.pt. Tel: +(351) 21 3925891. Fax: +(351) 21 3974153.

1 Introduction

Possibly, one of the most important recent contributions in economic theory is related to the integration of spatial processes within the aggregate dynamics and economic growth (see Krugman (1998) and (Lopes, 1980, p. 123-138) for an early perspective). The new economic geography has the potential for integrating spatial location, diffusion and agglomeration, studied by spatial economic, with country or regional specialization, studied by international economics, and growth and convergence among regions and countries, studied by growth theory.

Interestingly, some puzzles seem to emerge. In particular, convergence among countries is a stylized fact brought about by growth economics (see Barro and Sala-I-Martin (2004)), while spatial agglomeration, of both industries and overall economic activity, seem to emerge as a main topic of research to the new economic geography Fujita and Thisse (2002)).

In our view, the solution of this puzzle is call for a formal structure in which the dynamics across time and space is jointly determined. One possible reason for the difficulty in solving that puzzle is related to the fact that economic theory does not offer simple mechanisms for considering them jointly. The benchmark models in growth economics are a-spatial and the the benchmark models in the new economic geography paradigm tend to consider only two locations in space (see Fujita et al. (1999) and Fujita and Thisse (2002)).

There is some literature that presents models dealing jointly with time and space, by using partial differential equations (see Beckmann (1952), Beckmann (1970) and Isard and Liossatos (1979)). Recently this approach has been reintroduced by several authors dealing with several problems v.g. Lucas (2001), (Fujita and Thisse, 2002, ch. 9) and Mossay (2003).

The interaction between growth in time and space has been studied, within this framework, by Isard and Liossatos (1979), Brito (2004), which present a spatial extension of the Ramsey (1928) model, and Camacho and Zou (2004), which present a spatial extension of the Solow (1956) model. These papers consider a world composed by open regions, each one populated by a potentially heterogeneous household, and trading goods and capital. The time-space connection is modeled by assuming that capital flows in the opposite direction of the gradient of the distribution of capital across space.

In this paper we also consider a spatial version of the Solow (1956) model, by relying on the accounting framework presented in Brito (2004), which has been borrowed by Camacho and Zou (2004). We extend it in several ways. First, we assume that population varies through time and migrates across space and second, we consider a case in which there is unbounded growth. A balanced growth path (BGP) will exist. As our intention is to a large extent methodological, we will choose the most simple structure; in particular the space-time dynamics of labor is exogenous.

However, our approach allows for deal with several issues in the intersection of growth and geographical economics: the existence of diffusive and agglomeration forces and the relationship between $\beta-$ and $\sigma-$ convergence, in a context in which there is unbounded growth (though generated by exogenous variables). While diffusion is embedded in the setup of the model we will get agglomeration caused by exogenous heterogeneity in the parameters of the model, that is, we will not have agglomeration as produced endogenously by some externality. Depending on the assumptions about behavioral parameters, we will have convergence towards a symmetric (absolute convergence) or to asymmetric (relative convergence) BGP is a property of the distributional dynamics of the model. The distinction between, and the dynamic processes generating, β-convergence and $\sigma-$ convergence will also be transparent: diffusion, related to spatial convergence, will be the driving force behind $\sigma-$ convergence, while stability across time, will imply β-convergence.

Intuitively, the spatial equalization of the rates of return of capital, generate by capital flows across regions will bring about σ-convergence, while decreasing marginal returns to capital will generate stability across time and β-convergence. Permanent asymmetries in productivity will cause agglomeration (and relative not absolute convergence).

The rest of the paper contains the following sections. Section 2 presents the model, section 3 defines and determines that balanced growth path, section 4 studies the stability properties of the model, in a distributional sense, section 5 studies the dynamics of the adjustment towards the BGP when the initial distribution of capital and labor deviates from a homogeneous BGP, section 6 studies comparative distributional dynamics for an asymmetric productivity shock and section 7 concludes.

2 The model

All the variables are functions of the space-time pair $(x, t) \in \mathbb{R} \times \mathbb{R}_+$. We assume that there is a continuous of potentially heterogeneous agents, that interact among themselves, and are indexed by x. We identify the support of heterogeneity with space, and conceive the interspatial dynamics as being originated from that heterogeneity. Therefore, space should be understood as the support for a distribution of a variable that takes heterogeneous values cross sectionally. In particular, the stock of capital and population at location x, at time t, are denoted by $K = K(x, t)$ and $N = N(x, t)$.

This location device, as an indexing mechanism, was already considered by Hotelling (1929): we start from an initial distribution and index the locations in an descending order, and keep, subsequently, the same indexing. In order to avoid the introduction of any arbitrary kind of structure we assume that space is unbounded, and there are no conditions related to space, in particular, spatial boundary conditions. This allows for the existence of spatially homogeneous values for the variables, as a possible state of the world.

2.1 Location x

A household is located at every location x. They may be heterogeneous along several dimensions: as regards endowments of capital and labor or behavioral parameters (productivity, savings and technology). Each household performs production, consumption, and investment activities, denoted by $Y(x, t)$, $C(x, t)$ and $I(x, t)$, respectively, at every moment t.

Production uses capital and labor inputs, with a neoclassical technology:

$$Y(x,t) = F[K(x,t), A(x,t)N(x,t)],$$

where $F(.)$ is increasing in both arguments, it is concave, linearly homogeneous and displays the Inada properties. Technical progress is labor augmenting, which is a necessary condition for the existence of a balanced growth path.

The Euler theorem, for homogeneous functions, implies that the income generated from production is equal to the contributions of capital and labor,

$$Y(x,t) = r(x,t)K(x,t) + w(x,t)N(x,t),$$

where the rental rate of capital is denoted by $r(x,t) = \frac{\partial F(x,t)}{\partial K(x,t)}$ and the wage rate is $w(x,t) = \frac{\partial F(x,t)}{\partial N(x,t)}$.

We assume a keynesian ad-hoc consumption function, where consumption is a (potentially space dependent) proportion of income,

$$C(x,t) = (1 - s(x))Y(x,t),$$

where $0 < s(x) < 1$ for any $x \in \mathbb{R}$. Therefore, savings is $S(x,t) = Y(x,t) - C(x,t) = s(x)Y(x,t)$. The equilibrium condition, for location x, at time t, may be represented by the equality between savings and total investment

$$S(x,t) = I(x,t) + \tau(x,t),$$

where gross investment is the sum of net investment and capital depreciation, $I(x,t) = \frac{\partial K(x,t)}{\partial t} + \rho(x)K(x,t)$, where $0 < \rho(x) < 1$ and $\tau(x,t)$ is net asset investment in other locations. That total investment is equal to investment done in location x and in the other locations by an agent in location x. Both the savings and the rate of capital depreciation, $s(x)$ and $\rho(x)$, are assumed to be time-independent but (possibly) space specific.

2.2 Closed and open regions

In order to define equilibrium, we need consider regions instead of single locations. We distinguish between locations, $x \in \mathbb{R}$, and regions with width Δx_i, $X_i := [x_i, x_i + \Delta x_i)$, and assume that households are homogeneous within each region. The whole economy has the topology of a Borel space, such that $\mathbb{R} = \cup_i X_i$, is the union of mutually disjoint regions.

Then the (aggregate) equilibrium condition for region X_i is now

$$\int_{X_i} S(x,t) - I(x,t)dx = \int_{X_i} \tau(x,t)dx. \tag{1}$$

For the ensuing analysis, the distinction between closed and open regions is crucial.

If region X_i is *closed*, or autarkic, households will only invest in firms located within the same region, then $\int_{X_i} \tau(x,t)dx = 0$. In this case, as variables are homogeneous within regions, there is no interaction mechanism both within and among regions. The equilibrium values for all variables (v.g., $Y(.)$and $S(.)$), are monotonous functions of $K(.)$, whose dynamics is given by equation

$$\frac{dK(x,t)}{dt} = s(x)F\left(K(x,t), A(x,t)N(x,t)\right) - \rho(x)K(x,t), \;\; \forall (x,t) \in \mathbb{R} \times \mathbb{R}_{++} \tag{2}$$

given an initial distribution of capital $K(x,0) = K_0(x)$ at time $t = 0$. The equilibrium dynamics is represented by a particular replication in space of a dynamics similar to the Solow (1956) model. The distribution of capital may change across time, from the initial distribution $K_0(x)$, but it will do it as a result of independent savings and investment decisions, without capital mobility between regions. This is the reasoning behind the direct application of the Solow equation in growth empirics, for assessing convergence among countries and regions (see Barro and Sala-I-Martin (2004)): the distribution will change because each region is converging towards its own BGP (which may be symmetric or asymmetric).

If region X_i is *open* then $\int_{X_i} \tau(x,t)dx \neq 0$ and we have to specify the a spatial interaction among regions. In order to introduce it, note that $\int_{X_i} \tau(x,t)dx$ represents the current account for region X_i. As the equilibrium equation (1) is formally similar to the budget constraint for each region, then $\int_{X_i} \tau(x,t)dx$ also represents the flow of capital originated therein. If a region has a current account surplus (deficit), that is if $\int_{X_i} \tau(x,t)dx > 0$ (< 0), then it will export (import) capital in addition to (in order to finance) its accumulation of its own capital stock.

Following Isard and Liossatos (1979) and Brito (2004), we assume that the flow of capital originated from region X_i is equal to the difference in the gradient of the capital distribution measured at the borders of region X_i

$$\int_{X_i} \tau(x,t)dx = -\left[\frac{\partial K(x_i + \Delta_i, t)}{\partial x} - \frac{\partial K(x_i, t)}{\partial x}\right].$$

This equation has some implicit assumptions and some implications, that are worth unveiling. For a given level of investment, we see from equation () that the current account, for any region X_i, is a positive function of the aggregate capital stock for that region. If the current account is positive then, in order for the balance of payments constraint to hold, the region should export capital. We assume that the net flow of capital is symmetric to the local gradient of the aggregate capital distribution, measured in the boundaries with the two adjacent regions. That is, the flow of capital is equal to the symmetric of the net difference of its own stock of capital to those of the adjacent regions. If the capital markets are perfect, and if there are no adjustment costs or other spatially related frictions to capital movements, then this would be equivalent to buying shares issued by other regions to finance their investment.

That assumption implies, together with the existence of decreasing marginal returns to capital in production, that regions which are richer (poorer) in capital have a lower (higher) rate of return. Then, our assumption regarding capital flows, amounts to hypothesising that capital flows consistently with the differences among rates of return in space: poor regions will have a higher capital income and therefore will attract capital.

As

$$\frac{\partial K(x_i+\Delta_i,t)}{\partial x}-\frac{\partial K(x_i,t)}{\partial x}=\int_{X_i}\frac{\partial^2 K(x,t)}{\partial x^2}dx$$

then

$$\int_{X_i}\tau(x,t)dx=-\int_{X_i}\frac{\partial^2 K(x,t)}{\partial x^2}dx. \tag{3}$$

If we assume that $\Delta x_i \to 0$ then the equilibrium capital accumulation equation will be

$$\frac{\partial K}{\partial t}=\frac{\partial^2 K}{\partial x^2}+s(x)F\left(K(x,t),A(x,t)N(x,t)\right)-\rho(x)K(x,t),\ \ \forall (x,t)\in\mathbb{R}\times\mathbb{R}_{++} \tag{4}$$

given an initial distribution of capital, $K(x,0)=K_0(x)$. This equation should be compared with the autarkic equation for the dynamics of capital (2). Both equations determine a flow of distributions $\{K(x,t),\ (x,t)\in(\mathbb{R}\times\mathbb{R}_+)\}$, however, while the distributions evolve only as a consequence of a temporal dynamics in equation (2), now the distribution of capital involves both temporal and spatial dynamics, by a dispersion (or diffusion) mechanism. This dispersion effect has been highlighted recently as one of the main spatial dynamics mechanisms.

We will consider that regions are open from now on. Previous papers using a similar setup assume bounded growth (Isard and Liossatos (1979), Brito (2004), Camacho and Zou (2004)). In this paper we extend them by considering unbounded growth, generated from two sources: labor dynamics and technical progress.

2.3 Labour dynamics

As there is no unemployment, then the labor input equals population. The population is distributed across space [1], but grows at a common rate, $\mu \geq 0$, which is equal to the difference between fertility and death rates. However, the change in population at the rate $\mu N(x,t)$ does not translate in an equivalent change in population at location x, because we assume that there is migration. In order to concentrate upon the dynamics of capital accumulation, we assume that net migration is exogenous and negatively related to the gradient of population distribution across space. This is equivalent to assuming that population grows through time and diffuses across space according to the equation

$$\frac{\partial N(x,t)}{\partial t}=\frac{\partial^2 N(x,t)}{\partial x^2}+\mu N(x,t) \tag{5}$$

given $N(x,0)=N_0(x)$.

Though this equation generates unbounded population growth along time, we can separate it between unbounded and transient components, by setting

$$N(x,t)=n(x,t)e^{\mu t},\ \ \forall (x,t)\in\mathbb{R}\times\mathbb{R}_+. \tag{6}$$

[1] Of course we sidestep several problems, that are present in the literature, related to the fact that population does not live in $\mathbb{R}$, that population should be bounded, both as regards time and space,etc.

In order to get positive population in every point in space, and to have an homogeneous distribution of population across space as a particular asymptotic state, we define further

$$n(x,t) = \overline{n} + u_n(x,t), \quad \forall (x,t) \in \mathbb{R} \times \mathbb{R}_+,$$

where $\overline{n} > 0$ and is spatially homogenous. As $u_n(x,t) = N(x,t)e^{-\mu t} - \overline{n}$, then from equation (5) we get

$$\frac{\partial u_n(x,t)}{\partial t} = \frac{\partial^2 u_n(x,t)}{\partial x^2}. \tag{7}$$

Two alternative initial conditions may be considered: a spatially homogeneous case $u_n(x,0) = 0$ if $n(x,0) = \overline{n}$ or a spatially heterogeneous case $u_n(x,0) \neq 0$.

The solution of equation (7)[2] is

$$u_n(x,t) = \begin{cases} \int_{-\infty}^{\infty} u_n(x,0)\delta(x-\xi)d\xi, & t = 0, \\ \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} u_n(x,0)e^{-\frac{(x-\xi)^2}{4t}} d\xi, & t > 0 \end{cases} \tag{8}$$

Two properties regarding population dynamics may be derived from this equation

$$\begin{aligned} \lim_{t\to 0^+} u_n(x,t) &= u_n(x,0) \\ \lim_{t\to\infty} u_n(x,t) &= 0. \end{aligned}$$

Therefore,

$$\lim_{t\to\infty} N(x,t) = \lim_{t\to\infty} \overline{n}e^{\mu t},$$

i.e., population tends asymptotically to an unbounded homogeneous spatial distribution.

The particular solution depends on the initial distribution. If the initial distribution is spatially homogeneous, that is if $u_n(x,0) = 0$ then $u_n(x,t) = 0$, for every $(x,t) \in \mathbb{R} \times \mathbb{R}_+$, which implies that $N(x,t) = \overline{n}e^{\mu t}$.

For the case in which the initial distribution is spatially heterogeneous assume that we have an initial normal distribution $u_n(x,0) \sim N(0,\sigma_n^2)$, that is

$$u_n(x,0) = \frac{1}{\sqrt{2\pi}\sigma_n} e^{-\frac{x^2}{2\sigma_n^2}}.$$

In this case the solution (8) becomes

$$u_n(x,t) = \frac{e^{-\frac{x^2}{2\sigma_n^2(t)}}}{\sqrt{2\pi}\sigma_n(t)}, \tag{9}$$

that is, it follows a normal distribution with a time dependent variance $\sigma_n^2(t) = \sigma_n^2 + 2t$. This solution also verifies the property $\lim_{t\to\infty} \sigma_n^2(t) = +\infty$, i.e., tends asymptotically to an homogenous distribution.

[2]See the derivation in the appendix.

Figure 1 shows the spatial-time dynamics for an initial normal deviation from a spatially homogenous population distribution. It shows, in the left panel, the transient deviation for a detrended homogeneous distribution, $n() = u_n(.) + \overline{n}$ and the level of population $N(.)$, in the right panel.

Note that, from the concavity of the production function, the wage rate is higher (lower) in the locations with lower (higher) labor supply. Therefore, labor flows from locations with lower wages to locations with higher wages, which is consistent with any reasonable endogenous spatial arbitrage that we may consider.

———————— figure 1: fig1.pdf fig2.pdf

2.4 Technical progress

We have introduced exogenous labor augmenting technical progress in order to have a balanced growth path. We will assume that

$$A(x,t) = a(x)e^{\gamma t} \tag{10}$$

where $\gamma \geq 0$ is the long run rate of growth of technical progress and $a(x)$ represents short run productivity levels. These short run deviations may display idiosyncratic differences throughout space. In this case, there will be permanent differences in $A(x,t)$ across time and space.

We could consider, instead, another diffusion process for technical progress. However, in this case the consequences of technical progress on the dynamics of the distribution of the capital stock would not be formally different from $N(.)$. In particular, the differences in productivity will die out asymptotically. If equation (10) holds, then differences in productivity will be permanent, and we want to explore explicitly the consequences of this case capital accumulation.

3 The Balanced growth path

If we substitute equations (6) and (10) into equation (4), we get the parabolic partial differential equation which represents the space-time dynamics of capital accumulation for a world composed by open regions

$$\frac{\partial K}{\partial t} = \frac{\partial^2 K}{\partial x^2} + s(x)F\left(K(x,t), a(x)n(x,t)e^{(\mu+\gamma)t}\right) - \delta K, \qquad (x,t) \in \mathbb{R}\times\mathbb{R}_+. \tag{11}$$

Let us represent the stock of capital in levels as a product of trend and transition

$$K(x,t) = k(x,t)e^{(\gamma+\mu)t}, \ \ \forall(x,t). \tag{12}$$

As a implication of the assumptions about the growth of population and technical progress, the long run growth rate is spatially independent and as the production function is linearly homogeneous, then the transition dynamics is given by

$$\frac{\partial k}{\partial t} = \frac{\partial^2 k}{\partial x^2} + s(x)F\left(k(x,t), a(x)n(x,t)\right) - (\delta+\mu+\gamma)k(x,t) \tag{13}$$

where $k(x,0) = K(x,0)$ is given.

We will define the *balanced growth path* (BGP), denoted by $\{\overline{K}(x,t),\ \ (x,t) \in \mathbb{R} \times \mathbb{R}_+\}$, as the flow of distributions such that

$$\overline{K}(x,t) = \overline{k}(x)e^{(\gamma+\mu)t}, \ \ \forall (x,t) \in \mathbb{R} \times \mathbb{R}_+, \tag{14}$$

such that $\{\overline{k}(x),\ x \in \mathbb{R}\}$ solves

$$\frac{\partial^2 \overline{k}}{\partial x^2} + s(x)F\left(\overline{k}(x), a(x)\overline{n}\right) - (\delta + \mu + \gamma)\overline{k}(x) = 0, \ \ \forall x \in \mathbb{R},$$

where population is at their long run level and the distribution of capital is time-independent. The solution of this equation represents the asymptotic state of equation (13).

We will study the existence, unicity and the properties of this balanced growth path, both from the perspective of (distributional) stability and long run characteristics, for the particular case in which there is spatial homogeneity.

If savings and exogenous productivity are spatially homogeneous, $s(x) = s$ and $a(x) = a$ for all $x \in \mathbb{R}$ then

$$\overline{K}(x,t) = \overline{k}e^{(\gamma+\mu)t}, \ \ \forall (x,t) \in \mathbb{R} \times \mathbb{R}_+, \tag{15}$$

such that

$$\overline{k} = \{k : sF(k, a\overline{n}) = (\delta + \mu + \gamma)k\}.$$

As it is a replication of the well-know Solow (1956) and Swan (1956) through space, then we know that $\overline{k}$ exists and it is unique. The long run growth rate is equal to the sum of the rate of technical progress and of population growth. Then, along the BGP, all the locations are symmetric: the capital stock is the same and grows at the same rate.

Next we will characterize the local dynamics and perform a generalized comparative spatial-temporal dynamics exercise.

4 Local dynamics

In order to determine whether and how the distribution of capital evolves across time and space if the initial distribution deviates from a BGP distribution or if there is a spatially symmetric shock on any parameter we should solve equation (13). However, as it is non-linear and is only defined qualitatively, then we have to resort to qualitative stability analysis, by linearizing it in the in the neighborhood of a spatially homogeneous BGP.

Let us consider the variation at the neighborhood of the BGP as given by $K(x,t) - \overline{K} = u_k(x,t)e^{(\mu+\gamma)t}$ where

$$u_k(x,t) := k(x,t) - \overline{k}.$$

Then, the following variational PDE represents the dynamics in the neighborhood of a homogeneous BGP

$$\frac{\partial u_k}{\partial t} = \frac{\partial^2 u_k}{\partial x^2} + \lambda u_k(x,t) \tag{16}$$

where $\lambda := sr - (\delta + \mu + \gamma) < 0$, where $r = \frac{\partial F(k,an)}{\partial k}$ from the concavity of the production function.

After solving equation (16)[3], we get

$$u_k(x,t) = \begin{cases} \int_{-\infty}^{\infty} u_k(x,0)\delta(x-\xi)d\xi, & t = 0 \\ e^{\lambda t}\int_{-\infty}^{\infty} u_k(x,0)\frac{e^{-\frac{(x-\xi)^2}{4t}}}{\sqrt{4\pi t}}d\xi, & t > 0 \end{cases} \tag{17}$$

Note that $\lim_{t\to 0^+} u_k(x,t) = u_k(x,0) = k(x,0) = K(x,0)$. As, for a bounded initial distribution, the kernel is a normal distribution with a variance tending to infinity, them the second term in equation (17) will be bounded asymptotically. Then, the stability condition is $\lambda < 0$. This condition is met, as a consequence of the decreasing returns to capital. The introduction of spatial diffusion does not change this result when we compare to the a-spatial growth model.

Therefore, $\lim_{t\to\infty} u_k(x,t) = 0$ and we have what is termed in growth theory absolute β-convergence: first, if the behavioral parameters are symmetric across locations, differences in the growth rates only have a transitional nature, second, in the long run the GDP per capita is equalized, and, third, locations which have a smaller initial capital endowment will have higher transient growth rates, in the transition to a symmetric BGP.

Let us assume that the the initial distribution of capital is normal with zero mean and variance equal to σ_k^2. Then

$$u_k(x,t) = \frac{e^{-\frac{x^2}{2\sigma_k^2(t)}+\lambda t}}{\sqrt{2\pi}\sigma_k(t)} \tag{18}$$

follows a discounted normal distribution with variance $\sigma_k^2(t) = \sigma_k^2 + 2t$, with a discount factor equal to $e^{\lambda t}$. Then, there is σ-convergence: first, the variance of $u_k(.,t)$ increases along time, meaning that the capital stock is increasingly more evenly distributed across space and, second, as $\lim_{t\to\infty}\sigma_k^2(t) = +\infty$ the asymptotic distribution is spatially homogenous, for any initial variance of $u_k(.)$. Figure 2 depicts the solution for this case, for a particular choice of parameters and calibration. It clearly shows the convergence of the detrended variable, $k(x,t)$ towards a constant spatially homogeneous $\overline{k}$ and the convergence of the variable in levels $K(x,t)$ towards the a spatially homogeneous BGP $\overline{K}(x,t)$.

Summing up, we have both absolute β- and σ- convergence. However, we saw that there are two different mechanisms that bring them about: decreasing returns generate asymptotic stability and tend to dampen out initial differences in capital endowments (when there is spatial symmetry in all the parameters) and the diffusion mechanism, associated to the existence of perfect capital markets, tends to equalize capital distribution across space.

———- figure 2: : fig3.pdf fig4.pdf

5 Asymmetric population distributions

We assumed in the former section that labor was symmetrically distributed across space. In this section, two deviations from a homogeneous BGP are con-

[3]See the appendix.

sidered: first, as in the previous section, the initial distribution is not spatially homogenous and population is not only asymmetrically distributed across space, but also is subject to the dynamics resulting from equation (8).

Now the variational system becomes

$$\frac{\partial u_k}{\partial t} = \frac{\partial^2 u_k}{\partial x^2} + \lambda u_k(x,t) + swu_n(x,t) \tag{19}$$

where $w = \frac{\partial F(k,an)}{\partial w} > 0$ is the wage rate. This represents a perturbation of equation (16) by a non-autonomous term in time and space.

The solution for equation (19) is

$$\begin{aligned} u_k(x,t) &= \frac{e^{\lambda t}}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} u_k(x,0) e^{-\frac{(x-\xi)^2}{4t}} d\xi + \\ &+ sw \int_0^t e^{\lambda(t-\tau)} \int_{-\infty}^{\infty} u_n(\xi,0) \frac{e^{-\frac{(x-\xi)^2}{4t}}}{\sqrt{4\pi t}} d\xi d\tau, \quad t > 0. \end{aligned} \tag{20}$$

While the initial distribution of the capital stock, as regards a homogeneous BGP, affects the transient dynamics of capital accumulation by the present value of the initial deviation, the deviation of population as regards the long run trend will do it through the sum of the present values of the flow of deviations. The rate of discount is again $-\lambda$.

For normal initial distributions of both capital and population, the solution of equation (19) will be

$$u_k(x,t) = \frac{e^{\lambda t - \frac{x^2}{2\sigma_k^2(t)}}}{\sqrt{2\pi}\sigma_k(t)} + \frac{sw}{\sqrt{2\pi}\sigma_n(t)} \int_0^t e^{\lambda(t-\tau) - \frac{x^2}{2\sigma_n^2(t)}} d\tau, \quad t > 0, \tag{21}$$

where, again, $\sigma_n^2(t) = \sigma_n^2 + 2t$, and $\sigma_k^2(t) = \sigma_k^2 + 2t$.

Figure 3 represents the solution for particular values of the parameters, by showing k, K and $\frac{K}{N}$. We have assumed that $\sigma_k^2 > \sigma_n^2$, which implies that the initial ratio is smaller for the regions with both higher initial capital stock and population. Again, we have both absolute β- convergence and σ-convergence.

—— figure 3: fig5.pdf, fig6.pdf, fig7.pdf

6 Permanent productivity asymmetries

Let us assume that the economy is initially at a homogeneous BGP and assume that there is a productivity shock which is permanent, constant through time, but asymmetric across space, $u_a(x) = a(x) - \overline{a}$.

Now the variational system becomes

$$\frac{\partial u_k}{\partial t} = \frac{\partial^2 u_k}{\partial x^2} + \lambda u_k(x,t) + s\frac{wn}{a} u_a(x), \tag{22}$$

where $s\frac{wn}{a} > 0$ and constant. The solution for equation (22) [4] is

$$u_k(x,t) = s\frac{wn}{a} \int_0^t e^{\lambda(t-\tau)} \int_{-\infty}^{\infty} u_a(\xi) \frac{e^{-\frac{(x-\xi)^2}{4(t-\tau)}}}{\sqrt{4\pi(t-\tau)}} d\xi d\tau, \quad t > 0. \tag{23}$$

[4]See, again, the appendix for a proof.

The solution may be intuitively seen as the sum of the present values of a weighted distributions of permanent time-space deviations of productivity, in which the kernel of the distribution, for every moment in time, is the normal density with variance equal to $2t$.

If the asymmetric shock is normally distributed with variance σ_a^2, then the solution is

$$u_k(x,t) = s\frac{wn}{a}\int_0^t e^{\lambda(t-\tau)}\frac{e^{-\frac{x^2}{2\sigma_a^2(t-\tau)}}}{\sqrt{2\pi}\sigma_a(t-\tau)}d\tau \quad t>0 \tag{24}$$

where $\sigma_a^2(t) = \sigma_a^2 + 2t$. See figure for a geometrical representation.

Though we start from a homogenous BGP, a permanent asymmetric productivity shock will imply a permanent asymmetric shift in the BGP. That is, it will become spatially heterogeneous, though all locations will grow asymptotically at the same rate of growth, as figure 4 shows.

Several interesting properties are present in this case. First, we will have relative β-convergence not absolute β-convergence: all locations will converge towards their own specific BGP capital stock levels which will differ across space. Second, though we will also have σ-convergence, the the variance will not tend towards infinity. We may interpret this case as an instance of the existence of agglomeration. Third, there are some partial spillover effects from the regions in which the productivity shock is larger towards other regions, which are decreasing with distance. Those spillover effects will not imply a complete spatial dilution of the productivity shock, that is, the agglomeration effect, while decreasing along the transition towards the BGP, are not eliminated.

——————— figure 4

7 Concluding remarks

We think that the use of partial differential equations allow for a simple, yet rich formal framework for dealing with dynamics processes in time and space. The main advantage of that mathematical structure is that it deals directly with the dynamics of distributions across space. As we have tried to show, they allow for the integration of several processes which have been dealt separately in growth theory and spatial economics. In particular, the twin processes of convergence across countries with agglomeration in some regions within countries. As in this paper we have presented a simple extension of the a-spatial growth model, agglomeration effects may only emerge exogenously. Elsewhere (see Brito (2004)) we show that, if savings are endogenous agglomeration may be endogenously generated through a process similar to spatial pattern formation.

The main difficulty lies clearly in the crucial role of the spatial interactions mechanisms and in the lack of formal theories to model them.

References

BARRO, R. J. AND X. SALA-I-MARTIN (2004): *Economic Growth*, MIT Press, 2nd ed.

Beckmann, M. (1952): "A Continuous Model of Transportation," *Econometrica*, 20, 643–660.

Beckmann, M. J. (1970): "The analysis of spatial diffusion processes," *Papers of the Regional Science Association*, 25, 109–17.

Brito, P. (2004): "The dynamics of growth and distribution in a spatially heterogeneous world,"Working papers, Department of Economics, ISEG, WP13/2004/DE/UECE.

Camacho, C. and B. Zou (2004): "The spatial Solow model," *Economics Bulletin*, 18, 1–11.

Fujita, M., P. Krugman, and A. Venables (1999): *The Spatial Economy. Cities, Regions and International Trade*, MIT Press.

Fujita, M. and J.-F. Thisse (2002): "Agglomeration and Growth with Migration and Knowledge Externalities," .

Hotelling, H. (1929): "Stability in Competition," *Economic Journal*, 39, 41–57.

Isard, W. and P. Liossatos (1979): *Spatial Dynamics and Optimal Space-Time Development*, North-Holland.

Kammler, D. W. (2000): *A first course in Fourier Analysis*, Prentice-Hall.

Krugman, P. (1998): "Space: the last frontier," *Journal of Economic Perspectives*, 12, 161–174.

Lopes, A. S. (1980): *Desenvolvimento Regional. Volume I: Problemática, Teoria, Modelos*, Fundação Calouste Gulbenkian.

Lucas, R. E. (2001): "Externalities and Cities," *Review of Economic Dynamics*, 4, 245–74.

Mossay, P. (2003): "Increasing Returns and Heterogeneity in a Spatial Economy," *Regional Science and Urban Economics*, 33, 419–444.

Pinsky, M. A. (1998): *Partial Differential Equations and Bounday-value Problems with Applications*, Waveland.

Ramsey, F. (1928): "A mathematical theory of saving," *Economic Journal*, 38, 543–59.

Solow, R. (1956): "A contribution to the theory of economic growth," *Quarterly Journal of Economics*, 70, 65–94.

Swan, T. (1956): "Economic Growth and Capital Accumulation," *Economic Record*, 32, 334–61.

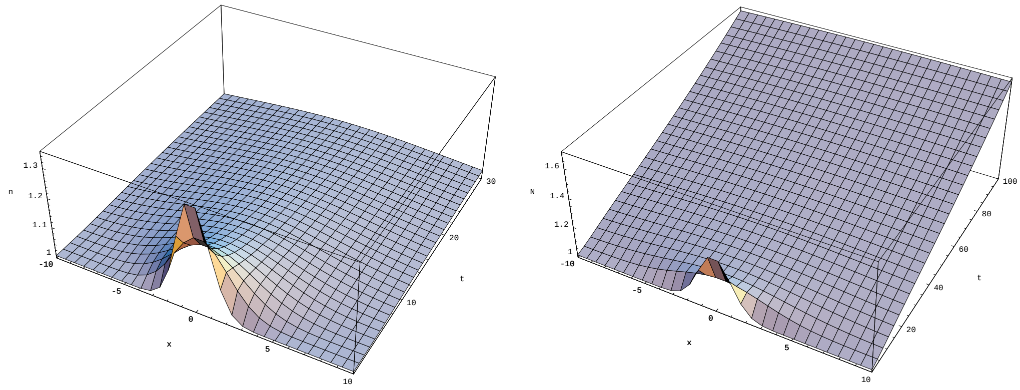

Figure 1: Labour dynamics: Solutions for $n(x,t) = u_n(x,t) + \overline{n}$ and $N(x,t) = n(x.t)e^{\mu t}$, for $\overline{n} = 1$, $\mu = 0.005$, and for $u_n(x,0) \sim N(0,1)$.

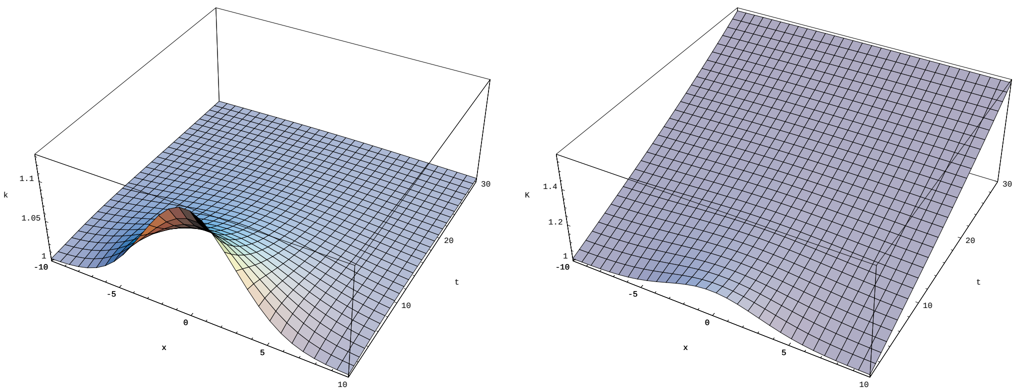

Figure 2: Stability: Solutions for $k(x,t) = u_k(x,t) + \overline{k}$ and $K(x,t) = k(x,t)e^{(\mu+\gamma)t}$, for a Cobb-Douglas production function, $F() = k^{\alpha}(an)^{1-\alpha}$ with capital share $\alpha = 0.3$, for $\overline{k} = \overline{n} = 1$, $\mu = 0.005$, $\gamma = 0.01$, $s = 0.2$ and $\rho = 0.1$. We found by calibration $a = 0.158086$. Then $\lambda \approx -0.0805$. The initial distribution is assumed to be $u_k(x,0) \sim N(0,3)$.

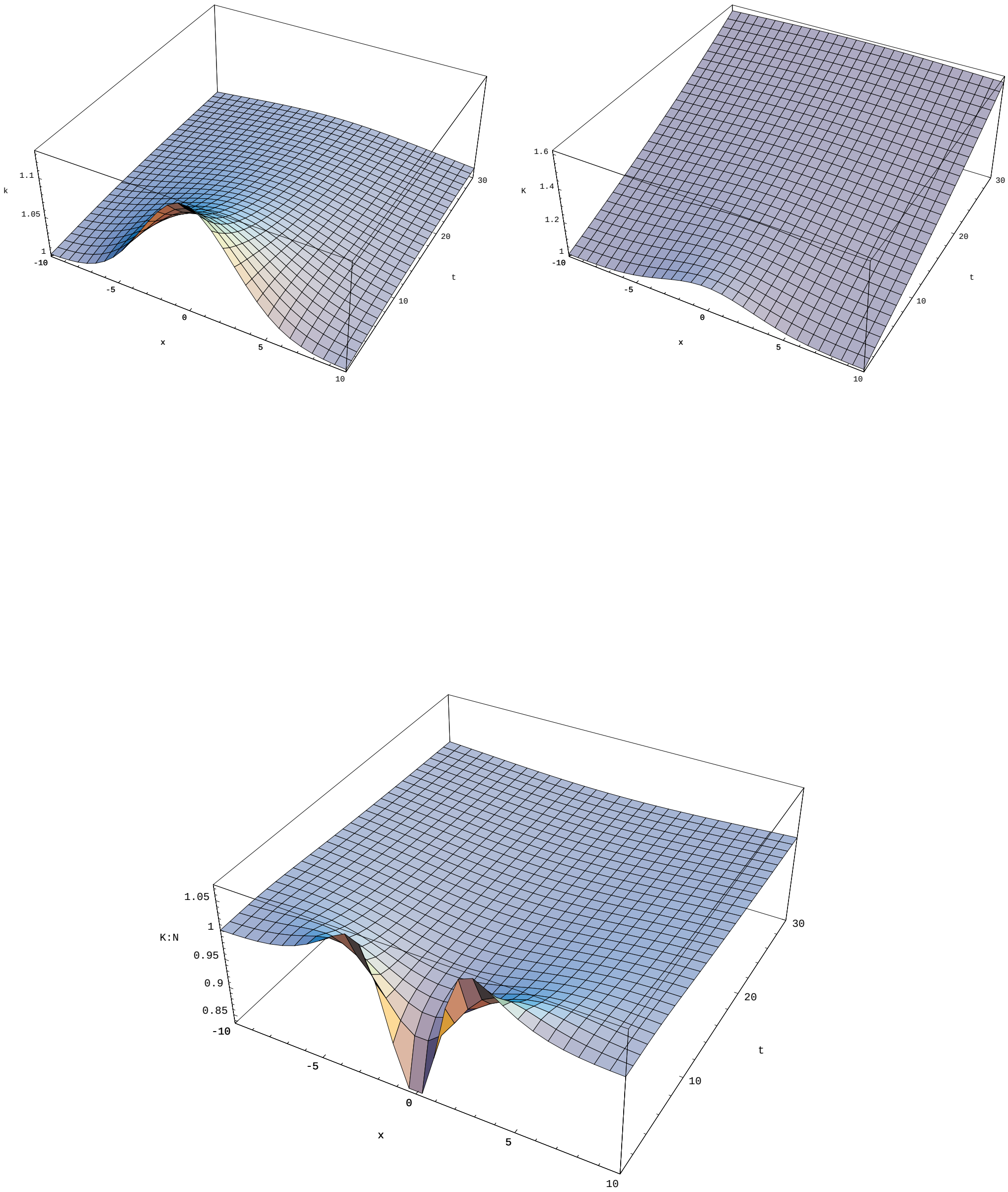

Figure 3: Capital accumulation for non-equilibrium labor supply: solutions for $k(x,t)$, $K(x,t)$ and $\frac{K(x,t)}{N(x,t)}$. Same values for the parameters as for figures 1 and 2.

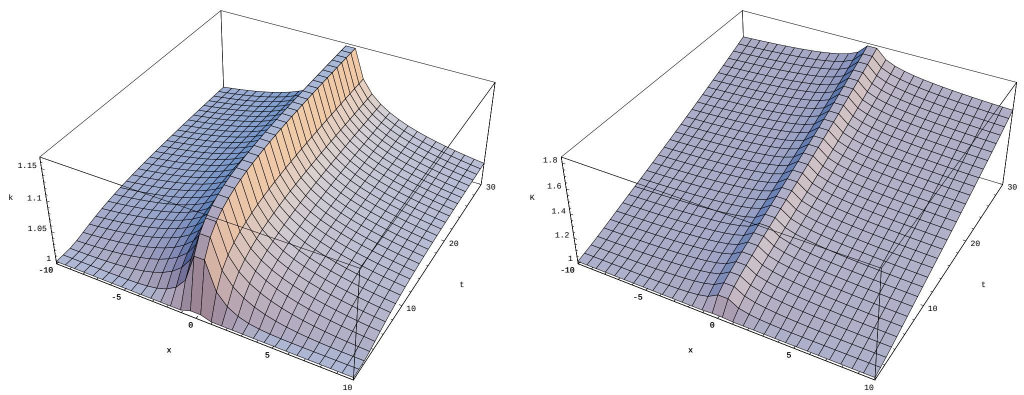

Figure 4: Effects of a permanent normal productivity shock: solutions for $k(x,t)$ and $K(x,t)$ for $k(x,0) = \overline{k} = 1$ and for $u_a(x) = a(x) - a \sim N(0, 0.05)$

A Proofs

Proof of equation (8): Equation (7) is a well know parabolic partial differential equation (PDE), the heath equation. Though the solution is well know, we will present it both for the convenience of the reader and for building upon it when determining the solution for the ensuing PDE's.

There are several solution methods for parabolic PDE's. As space is unbounded, we will apply Fourier transforms (see Kammler (2000) and Pinsky (1998)). Fourier transforms allows to define a pair of functions $u(x,t)$ and $U(\omega,t)$, if $\int_{-\infty}^{\infty} \mid u(x,t) \mid dx < \infty$, by $F[u(x,t)](\omega) = U(\omega,t)$ and $u(x,t) = F^{-1}[U(\omega,t)](x)$ where

$$U(\omega,t) = \int_{-\infty}^{\infty} e^{-2\pi i\omega x} u(x,t)dx$$

and

$$u(x,t) = \int_{-\infty}^{\infty} e^{2\pi i\omega x} U(\omega,t)dx$$

If we apply the Fourier transform to equation (7), then we get an ordinary differential equation (ODE)

$$\frac{\partial U_n(\omega,t)}{\partial t} = (2\pi i\omega)^2 \partial U_n(\omega,t)$$

given the initial value $U_n(\omega,0) = F(u_n(x,0))$. After noting that $i^2 = -1$, the solution of the ODE is

$$U_n(\omega,t) = U_n(\omega,0)G(\omega,t)$$

where $G(.)$ is the gaussian factor

$$G(\omega,t) := e^{-4\pi^2\omega^2 t}$$

As the inverse Fourier transform of a product of two Fourier transforms is a convolution, then the general solution of equation (7) is

$$\begin{aligned} u_n(x,t) &= u_n(x,0) * g(x,t) \\ &= \int_{-\infty}^{\infty} u_n(x^{'},0)g(x-x^{'},t)dx^{'} \end{aligned} \tag{25}$$

where the (heath) kernel of the inverse Fourier transform of the gaussian factor, $g(x,t) = F^{-1}(G(\omega,t))$, is

$$g(x,t) = \begin{cases} \delta(x), & t = 0 \\ \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}, & t > 0. \end{cases} \tag{26}$$

where $\delta(.)$ is the Dirac's delta function. Then, equation (8) is easily obtained.

Proof of equation (9): When the initial condition is represented by a normal distribution, then we have a convolution of two normal densities, in equation (25), because the kernel $g(.)$, in equation (26) is also a normal density.

As the convolution of two normal densities is a normal density whose mean is the sum of the two means (both equal to zero, in our case) and the variance is the sum of the two variances (σ_n^2 and $2t$), then we get equation (9) as a normal density $N(0, \sigma_n^2 + 2t)$.

Proof of equation (17) and (18) : Consider the following variable $v(x,t) := e^{-\lambda t} u_k(x,t)$. Then, we get the following PDE

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2}$$

which is similar to equation (8). Therefore, we can use the same procedure to get $v(x,t) = v(x,0) * g(x,t)$ and $u_k(x,t) = e^{\lambda t}[u_k(x,0) * g(x,t)]$, because $v(x,0) = u_k(x,0)$. Therefore

$$u_k(x,t) = e^{\lambda t} \int_{-\infty}^{\infty} u_k(x',0) g(x - x', t) dx'. \qquad (27)$$

where the kernel is given by equation (26). Then equation (17) is easily obtained. We get equation (18) by using the same procedure as for equation (9): $v(x,t) = u_k(x,0) * g(x,t)$ is the convolution of two normal distributions and is therefore a normal density $N(0, \sigma_k^2 + 2t)$.

Proof of equation (20): If we perform the same transformation as before, then equation (19) is equivalent to

$$\frac{\partial v(x,t)}{\partial t} = \frac{\partial^2 v(x,t)}{\partial x^2} + \beta(x,t) \qquad (28)$$

where $v(x,t) := e^{-\lambda t} u_k(x,t)$ and $\beta(x,t) := e^{-\lambda t} swu_n(x,t)$, given $v(x,0) = u_k(x,0)$. By using Fourier transforms we get the ODE

$$\frac{\partial V(\omega,t)}{\partial t} = -(2\pi\omega)^2 V(\omega,t) + B(\omega,t)$$

where $B(\omega,t) = F[\beta(x,t)](\omega)$, which has the solution

$$V(\omega,t) = V(\omega,0)G(\omega,t) + \int_0^t B(\omega,s)G(\omega,t-s)ds$$

were $G(\omega,\tau)$ is the gaussian factor. If we use the inverse Fourier transforms, we get

$$v(x,t) = v(x,0) * g(x,t) + \int_0^t \beta(x,\tau) * g(x,t-\tau)d\tau$$

but

$$\begin{aligned}
\int_0^t \beta(x,s) * g(x,t-\tau)d\tau &= \int_0^t e^{-\lambda\tau} swu_n(x,\tau) * g(x,t-\tau)d\tau = \\
&= sw \int_0^t e^{-\lambda\tau} u_n(x,0) * g(x,\tau) * g(x,t-\tau)d\tau = \\
&= sw \int_0^t e^{-\lambda\tau} u_n(x,0) * g(x,t)d\tau
\end{aligned}$$

because $g(x,t)$ is a normal density with variance equal to $2t$. Then

$$\begin{aligned} u_k(x,t) &= e^{\lambda t} v(x,t) = \\ &= e^{\lambda t} \int_{-\infty}^{\infty} u_k(\xi,0) g(x-\xi,t) d\xi + \\ &= +sw \int_0^t e^{\lambda(t-\tau)} \int_{-\infty}^{\infty} u_n(\xi,0) g(x-\xi,t) d\xi d\tau. \end{aligned}$$

Equation (20) results, and equation (21) is again obtained from the properties of the convolution of gaussian distributions.

Proof of equation (23): The application of Fourier transforms to equation (22) produces a linear parabolic PDE similar to (28) where $\beta(x,t) = e^{-\lambda t} s \frac{wn}{a} u_a(x)$, and, where $v(k,0) = u_k(x,0)$ as we assume that the economy is initially in a symmetric BGP. In this case the we get

$$\begin{aligned} u_k(x,t) &= e^{\lambda t} \int_0^t \beta(x,\tau) * g(x,t-\tau) d\tau \\ &= s \frac{wn}{a} \int_0^t e^{\lambda(t-\tau)} u_a(x) * g(x,t-\tau) d\tau \\ &= s \frac{wn}{a} \int_0^t e^{\lambda(t-\tau)} \int_{-\infty}^{\infty} u_a(\xi) g(x-\xi,t-\tau) d\xi d\tau, \end{aligned}$$

and, therefore, equation (23) results.