A Wavelet Exploration of the BVL Index

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Abstract

Wavelets allow for a more flexible characterization of time series than both spectral and classical time series methods, by representing them with basis functions that separate between time and scale. In this paper, we briefly present the main definitions and results of both wavelet and wavelet packet analyzes and apply some of them to the Lisbon stock exchange index (IBVL). We denoise the series, separate trend and cycle, and present a Monte Carlo estimation of the fractal dimension of the series.

Keywords: Wavelets; Lisbon stock exchange.

JEL Classification: C1, G1.

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1 Introduction

Wavelets and wavelet analysis are presenting an expanding body of theory, methods and algorithms that are being influential in signal processing, numerical analysis and mathematical modelling involving expansion, compression, description and smoothing of non-stationary time series.

There are several methods for representing time series (or functions of time). Classical time series methods represent them in the time domain, Fourier analysis in the frequency domain and short-run Fourier analysis and Wigner-Ville distributions in both time and frequency domains. Wavelets, represent time series in a time-scale domain.

Each type of representation involves the spanning of the original space by a particular set of functions, or basis functions. Wavelets (or small waves) allow for an accurate and flexible representation of signals by a small number of basis functions, endowed with time location properties. In this sense wavelet analysis generalizes Fourier analysis, by allowing a local representation of frequencies.

Multiresolution is the central method in wavelet analysis. The general idea consists in expanding a function, or a time series, in terms of levels of resolution, or detail, starting from an initial coarser representation. The expansion, by wavelet transformation, uses two basis-function spaces, the scaling and the wavelet functions. The first representing low frequencies and the latter high frequencies or detail.

There are several basis functions for which we can build a wavelet system (Haar, tent, spline, Daubechies, Coifflets). One important property of a wavelet system is that it allows for the expansion of any function or time series by a small number of factors (this implies that wavelets form unconditional bases). A wavelet transform involves a sparse matrix representation of a function, or signal, which is a property that is particularly useful in numerical analysis (see the papers in Dahmen et al. (1997) on numerical solutions for partial differential equations).

Wavelet analysis is being applied to an increasing number of fields: signal processing, seismology, pattern detection and numerical analysis, for instance. In particular, there are applications of wavelet analysis for denoising, compressing, representing in the time-frequency domain and detecting singularities in non-stationary data. They are starting to be used in economics and finance: on the separation between trends and cycles (see Ramsey & Zhang

(1998) and Ramsey & Lampart (1998)), on the representation of short run volatility in markets (see Davidson et al. (1998) for raw material world markets) and on the estimation of the fractional exponent in ARFIMA models (see Jensen (1999)).

In this paper, we apply several wavelet methods to the Lisbon Stock exchange daily index (BVL), for a period ranging from 1988 to the end of 1997. In particular, we deal with the separation between trend and cycle in the index, by using several different wavelet methods, and present an estimate for the empirical density of the fractional parameters, by using Monte-Carlo experiments. As the wavelet methods are far from mainstream time series methods, our results should be seen as exploratory.

The paper unfolds as follows. In the second section we present the basic definitions and results of wavelet analysis: multiresolution analysis, which is the central method of wavelet analysis, denoising by wavelet thresholding, and the discrete wavelet transforms. In the third section we apply them to the BVL index.

2 Wavelet analysis

We start by presenting the basic method of wavelet analysis: multiresolution (or multiscale) analysis. In the second subsection we present a particular implementation of a wavelet system design for the Daubechies wavelet. In the third subsection we address the wavelet expansion of functions and the definition of the wavelet transform. In the fourth subsection, we briefly present a generalization of the wavelet packet analysis. The last subsection deals with the uses of thresholding methods for denoising time series [1].

2.1 Multiresolution analysis

Consider a function belonging to the space of square-integrable functions $f(t) \in L^2(\mathbb{R})$. Let $f_j(t) \in \mathcal{V}_j \subset L^2(\mathbb{R})$, where $j \in \mathbb{Z}$ refers to a given level of resolution. A *multiresolution analysis* of $L^2(\mathbb{R})$ consists in forming a sequence of subspaces

$$\ldots \subset \mathcal{V}_{-2} \subset \mathcal{V}_{-1} \subset \mathcal{V}_0 \subset \mathcal{V}_1 \subset \mathcal{V}_2 \subset \ldots$$

such that

[1] A recent introduction reference for wavelets is Burrus et al. (1998), which you should consult for details and references.

1 $\bigcap_{j\in\mathbb{Z}} \mathcal{V}_j = \emptyset$, and $\bigcup_{j\in\mathbb{Z}} \mathcal{V}_j = L^2(\mathbb{R})$.

2 For any $f \in L^2(\mathbb{R})$, $f(t) \in \mathcal{V}_j$ if and only if $f(2t) \in \mathcal{V}_{j+1}$.

3 For any $f \in L^2(\mathbb{R})$ and any $k \in \mathbb{Z}$, $f(t) \in \mathcal{V}_j$ if and only if $f(t-k) \in \mathcal{V}_j$.

4 There exists a scaling function $\varphi \in \mathcal{V}_j$ such that $\{\varphi(t-k)\}_{k\in\mathbb{Z}}$ is an orthogonal basis of $\mathcal{V}_j$.

4' There exists a scaling function $\varphi \in \mathcal{V}_j$ such that $\{\varphi(t-k)\}_{k\in\mathbb{Z}}$ is an orthonormal basis of $\mathcal{V}_j$.

An associated sequence of subspaces

$$\ldots \subset \mathcal{W}_{-2} \subset \mathcal{W}_{-1} \subset \mathcal{W}_0 \subset \mathcal{W}_1 \subset \mathcal{W}_2 \subset \ldots$$

exists such that

$$\mathcal{V}_{j+1} = \mathcal{V}_j \bigoplus \mathcal{W}_j,$$

that is, $\mathcal{W}_j$ is the orthogonal complement of $\mathcal{V}_j$ in $\mathcal{V}_{j+1}$. Intuitively, if $d_j(t) \in \mathcal{W}_j$ represents the details for the level of resolution j then two adjacent levels of resolution are related as $f_{j+1}(t) = f_j(t) + d_j(t)$. Then, the subspace $\mathcal{V}_j$ can be written as a direct sum of spaces $\mathcal{W}_{j'}$, as

$$\mathcal{V}_{j'} = \bigoplus_{j=0}^{j'} \mathcal{W}_j.$$

We call $\varphi_j(.) \in \mathcal{V}_j$ a *scaling function* and $\psi_j(.) \in \mathcal{W}_j$ a *wavelet.*

There are two families of functions $\{\varphi_{j,k}(t) := 2^{-\frac{j}{2}}\varphi(2^{-j}t-k)\}_{k\in\mathbb{Z}}$ and $\{\psi_{j,k}(t) := 2^{-\frac{j}{2}}\psi(2^{-j}t-k)\}_{k\in\mathbb{Z}}$ which form orthonormal bases of, respectively, $\mathcal{V}_j$ and $\mathcal{W}_j$.

Then, from assumptions 1, 2, 3 and 4' the scaling and the wavelet functions may be expanded as linear combinations of the basis functions of $\mathcal{V}_{-1}$ and $\mathcal{W}_{-1}$, respectively, as

$$\varphi(t) = \sqrt{2}\sum_{k=0}^{L_f-1} h_k \varphi(2t-k). \tag{1}$$

and

$$\psi(t) = \sqrt{2}\sum_{k=0}^{L_f-1} g_k \psi(2t-k). \tag{2}$$

where the coefficients $H := \{h_k\}_{k=1}^{L_f}$ and $G := \{g_k\}_{k=1}^{L_f}$ are the filters of length L_f. Compactly supported wavelets can be obtained in two instances: (a) if the sums (1) and (2) are finite; (b) if L_f is finite.

The basis functions have, or can be defined to have, several interesting properties. In particular the basis functions $\psi_{j,k}(t)$ are chosen to be orthogonal to low degree polynomials, which implies that $\psi(.)$ has M vanishing moments,

$$\int_{-\infty}^{+\infty} \psi(t) t^m dt = 0, \qquad 0 \leq m \leq M-1.$$

The wavelet basis filters H and G are related by $g_k = (-1)^k h_{L_f-k-1}$, for $k = 0, \ldots, L_f - 1$, where the number of filters L_f is related to the number of vanishing moments. The filter G has also M vanishing moments.

In practical implementations, the multiresolution analysis reduces to the choice of the filter H, i.e., to the functions h_k and g_k. The functions φ and ψ are not actually computed. In addition, setting the 'finest' scale as $j = J$ and the 'coarsest' scale as $j = 0$, the sequence of spaces becomes

$$\mathcal{V}_J \subset \mathcal{V}_{J-1} \subset \ldots \subset \mathcal{V}_0,$$

where $\mathcal{V}_0$ is finite-dimensional.

2.2 Wavelet system design

The coefficients in the basic recursion equation (**??**cal) must satisfy the two following conditions:

$$\begin{aligned} \sum_{k=0}^{L_f-1} h_k &= \sqrt{2} \\ \sum_{k=0}^{L_f-1} h_k h_{k+2m} &= \delta(m). \end{aligned}$$

The Daubechies method for constructing the basis function, for orthonormal wavelets with compact support with the maximum number of vanishing moments ($K = M - 1$), leads to the following theorem (see Burrus et al. (1998, p.76)):

Theorem 1. *The discrete-time Fourier transform of h_k, having K zeros at $\omega = \pi$, of the form*

$$H(\omega) = \left(\frac{1+e^{i\omega}}{2}\right)^K \mathcal{L}(\omega)$$

Table 1: Daubechies (4) scaling function and wavelet coefficients and their moments

n	$h(n)$	$g(n)$	$\mu_\varphi(k)$	$\mu_\psi(k)$	k
0	0.482963	0.129410	1.414213	0	0
1	0.836516	0.224144	0.896575	0	1
2	0.224144	-.836516	0.568406	1.224744	2
3	-0.129410	0.482963	-0.864390	6.6572012	3

satisfies $\mid H(\omega) \mid^2 + \mid H(\omega+\pi) \mid^2$ *if and only if* $L(\omega) = \mid \mathcal{L} \mid^2 = P(\sin^2(\frac{\omega}{2}))$ *with* $K \leq \frac{L_f}{2}$ *where*

$$P(y) = \sum_{k=0}^{K-1} \begin{pmatrix} K-1+k \\ k \end{pmatrix} y^k + y^K R\left(\frac{1}{2} - y\right)$$

and $R(y)$ *is an odd polynomial chosen so that* $P(y) \geq 0$ *for* $0 \leq y \leq 1$*.*

For a filter of length four, $L_f = 4$, the Daubechies scaling functions and wavelet coefficients, and their moments of order k, are given in Table 1

After the determination of the scaling and of the wavelet coefficients (which are actually used in applications) we may determine the scaling functions and the wavelets, using equations (1) and (2). For instance, for the scaling function, if we take the Fourier transform of $\varphi(.)$ and use the Fourier transform of h_k,

$$\Phi(\omega) = \int_{-\infty}^{+\infty} \varphi(t) e^{-i\omega t} dt \quad (3)$$

$$H(\omega) = \sum_{k=-\infty}^{+\infty} h_k e^{-i\omega k} \quad (4)$$

where $i = \sqrt{-1}$ and $k \in \mathbb{Z}$. If those Fourier transforms exist, then (3) can be written as $\Phi(\omega) = \frac{1}{\sqrt{2}} H(\frac{\omega}{2}) \Phi(\frac{\omega}{2})$, which upon iteration becomes

$$\Phi(\omega) = \prod_{k=1}^{\infty} \left\{ \frac{1}{\sqrt{2}} H\left(\frac{\omega}{2^k}\right) \right\} \Phi(0).$$

The wavelet may be determined by using the same method. We get the functions $\varphi(.)$ and ψ after applying the inverse Fourier transform. The Daubechies(4) scaling and wavelet functions are depicted in Figures 1 and 2, which also show their translation and dilation.

`[Figures 1 and 2 around here]`

2.3 Representation of functions in wavelet bases

Let $f(t) \in L^2(\mathbb{R})$ be any square-integrable function. Its projection onto the $\mathcal{V}_j$ subspace is given by

$$f_j(t) := (P_j f)(t) = \sum_{k\in\mathbb{Z}} f_k^j \varphi_{j,k}(t),$$

where P_j is the projection operator onto that subspace. The coefficients $\{f_k^j\}_{k\in\mathbb{Z}}$ are computed from the inner product

$$f_k^j = \int_{-\infty}^{+\infty} f(t)\varphi_{j,k}(t)dt.$$

From the former results, we can also write $(P_j f)(t)$ as the sum of the projections of $f(t)$ onto the subspaces $\mathcal{W}_l$, $l > j$, as

$$(P_j f)(t) = \sum_{l>j}\sum_{k\in\mathbb{Z}} d_k^l \psi_{l,k}(t).$$

where the coefficients $\{d_k^j\}_{k\in\mathbb{Z}}$, computed as

$$d_k^j = \int_{-\infty}^{+\infty} f(t)\psi_{j,k}(t)dt$$

are called the *discrete wavelet transforms* (DWT) of $f(t)$. The projection onto space $\mathcal{W}_j$ is denoted by $(Q_j f)(t)$, where $Q_j = P_{j-1} - P_j$.

The wavelet expansion system on the 'finest' subspace, i.e., the expansion into the wavelet basis of $(P_J f)(t)$, is given by a sum of successive projections on subspaces $\mathcal{W}_j$, $j = 1, 2, \ldots, J$ and a final 'coarse' projection on $\mathcal{V}_J$,

$$(P_J f)(t) = \sum_{j=1}^{J} \sum_{k\in\mathbb{Z}/2^{n-j+J}} d_k^j \psi_{j,k}(t) + \sum_{k\in\mathbb{Z}/2^{n-j+J}} f_k^0 \varphi_{0,k}(t),$$

where $n = \{x : \dim(\mathcal{V}_J) = N = 2^x\}$. Given the set of coefficients $\{f_k^0\}_{k\in\mathbb{Z}/2^{n-j}}$, which are the coefficients of the projection of $f(t)$ on $\mathcal{V}_0$, we use the wavelet bases expansions in the two projections of f to determine recursively

$$f_k^j = \sum_{l=1}^{L_f-1} h_l f_{l+2k+1}^{j-1}$$

$$d_k^j = \sum_{l=1}^{L_f-1} g_l f_{l+2k+1}^{j-1}$$

where $j = 1, 2, \ldots, J$ and $k \in \mathbb{Z}/2^{n-j+J}$, that is, $\{0, 1, \ldots, 2^{n-j} - 1\}$.

In practical implementations, the Mallat's algorithm is employed. It consists in, given a scale, going down to the next finer scale, by using the relation

$$f_{t+1}(k) = \sum_{m=0}^{L_f-1} f_j(m)h(k-2m) + \sum_{m=0}^{L_f-1} d_j(m)g(k-2m).$$

This amounts to calculating the DWT by using two filters: a low-pass (h) and a high-pass (g) filter. Also, the initial function may be recovered by computing the *inverse wavelet transform* (IDWT). A time series may be filtered if the inverse transformation is computed with a filter bank of smaller dimension than J.

2.4 Wavelet packet analysis

The former wavelet system may be generalized in several ways. A particular flexible generalization is the *wavelet packet analysis*. It allows for a flexible choice of the basis functions by operating further decompositions on them. At each level of resolution both the scaling function and the wavelet are further expanded.

For instance, the space $\mathcal{V}_2$ may be expanded as

$$\mathcal{V}_2 = \mathcal{V}_0 \bigoplus \mathcal{W}_0 \bigoplus_{i=0}^{1} \mathcal{W}_{1i} \bigoplus_{1}^{i=0} \bigoplus_{1}^{j=0} \mathcal{W}_{2ij}$$

where the first index for $\mathcal{W}$ stands for the level of resolution and the number and order of indexes 0 and 1 are related to the number and order of successive scaling function and wavelet decompositions, for that level of resolution.

From this perspective, the wavelet system is a special case in the sense that it has a particular base. As the number of bases can *a priori* be enormous, Coifman and Wickenhauser (see Coifman & Wickerhauser (1992)) developed an algorithm for choosing the *best basis*, which minimizes an entropy function.

2.5 Denoising

There are several methods for denoising a time series. Filtering based upon spectral methods involves computing the spectrogram by applying a discrete Fourier transform, passing a low-pass filter, after defining a particular maximum frequency, and going back to the time domain.

Denoising by using wavelet methods uses a similar approach in which the degree of smoothness is controlled by a *thresholding* over the wavelet transform of a series.

Denote a noisy series by $\{y_t\}_{t=1}^{N}$. Let it be represented by a sum of an unknown smooth trend and an additive noise component (cycle plus noise), as

$$y_t = x_t + \varepsilon u_i, \;\; u_i \sim \mathcal{N}(0,1) \;\text{ i.i.d. }\; i = 1, \ldots, N.$$

Let the wavelet transformation, in its own domain, be denoted as $Y = X+U$. In the literature, there are two main alternative thresholdings, in the wavelet domain: *hard thresholding* if

$$\hat{X} = T_h(Y,t) := \begin{cases} Y, & \mid Y \mid \geq t \\ 0, & \mid Y \mid < t, \end{cases}$$

and *soft thresholding* if

$$\hat{X} = T_s(Y,t) := \begin{cases} \text{sign}\{Y\}(\mid Y \mid -t), & \mid Y \mid \geq t \\ 0, & \mid Y \mid < t. \end{cases}$$

If we compute the IDWT then we get an estimate for x, $\hat{x}$. One of the good properties of the soft thresholding, as regards both hard-thresholding and Fourier-based methods, is that it keeps the smoothness properties of x, thus reducing the width of the spurious oscillations on u.

3 Application to the IBVL

Next we apply some of the concepts related to multiresolution analysis to the IBVL index, the Lisbon stock exchange index. We will use daily data from the first period in which the index was computed, January 2nd 1988, until the end of 1997. The sample has 2390 observations [2].

A first practical problem is related to the fact that 2390 is not dyadic, that is, there is no $x \in \mathbb{Z}$ such that it equals 2^x. Two solutions to this problem may be adopted. We may trim the series to $2^{11} = 2048$ observations and then "unpad" after getting the inverse wavelet transforms. One possible way for "unpadding" is just to fill out with zeros, but this will generate an undesirable discontinuity in the edges of the series. An alternative possibility,

[2] The software that we used, and in certain cases, adapted, was the following: `Gauss` Tsm toolbox, `Mathematica` Wavelet Toolbox, Donoho's et all Wavelab for `Matlab`.

which is suggested in the literature, is to expand the series in the following way. If the series is $\{x(t)\}_{t=1}^{T}$, with $T \neq 2^n$ for any $n \in \mathbb{Z}$, then the new series will be $\{x'(t)\}_{t=1}^{T'}$, such that

$$\ldots x_3, x_2, x_1, x_1, x_2, x_3, \ldots, x_{T-2}, x_{T-1} x_T, x_T, x_{T-1}, x_{T-2} \cdots$$

Next we apply several methods for separating the trend and the cycle, or, in other words, for denoising the series.

3.1 Denoising

We employ both the hard- and the soft-thresholding methods by choosing a normalized threshold, $t = \sqrt{2 \log T}$, and using the Daubechies(4) wavelet. Two observations should be made at this point: first, the length of the filter is inversely related to the smoothness of the trend, and, second, the trend in the two extremes of the time series is related to the unpadding (or cutting) method that we use in order to get the dyadic number. In this particular case, the detrended series is smaller than the noisy time series.

Basically our method follows four steps: first, we construct a dyadic series, second, we determine the wavelet transform and the wavelet packet transform by using the Daubechies(4) as low- and high-pass filters, third, we apply the thresholding definition to the wavelet transformed data, and, finally, we apply the inverse wavelet transform and the inverse wavelet packet transform to go back to the original space.

[Figure 3 around here]

Figure 3 and 6 present, in the left panel, the BVL index and in the right panel two denoised series, from wavelet and wavelet packet analyzes, respectively. Several observations can be drawn from those figures. First, the denoised series, irrespective of the denoising method, define a common pattern: there is a downward slopping trend until the mid of 1992 which is followed by an upward slopping trend, and, the increase in the last semester of 1997 is particularly strong thus defining, perhaps, a third phase. Second, soft thresholding produces smoother trends.

[Figures 4 to 6 around here]

Figure 4 displays the wavelet coefficients for the actual and the two denoised series. The noise is basically excluded by setting to zero the wavelet coefficients related to the lower levels of resolution and keeping the most significant coefficients for the higher levels of resolution [3].

Figure 7 presents an alternative method for denoising, based upon the choice of the most significant scales by using a maximum energy criterion. From the scalogram in the northeastern panel we see that most of the energy is concentrated in the scaling function (or trend). The third level is the one for which there is a maximum level of energy. Therefore, the smoothing out of the series uses the scaling function and the third level wavelet. The decomposition of the series between the trend and the cycle is shown in the lower panels. The resulting trend is very similar to the one which is obtained by hard thresholding. Though the series exhibits roughly an upward trend, note that the cycle data is clearly trendless.

`[Figure 7 around here]`

Two features of the BVL index may explain the similarity of the results that we get after applying different denoising procedures: the series is non-stationary and exhibits a decaying spectrum for the higher frequencies (see figure 5).

This may also be seen in figure 8, which presents a scalogram analysis for the daily rate of return. Now, as the trend is explicitly removed by first log differentiation, we get the smooth trend component by taking only the finer scale (i.e., the 11th level). Again, as in the series for the levels, note that some of the qualitative properties of the smoothed out trends seem to hold. However, the hike in 1997 is not present.

`[Figure 8 around here]`

Figure 9 displays a kernel estimation of the rates of return for 1, 2, 5, 10, and 30 days (or adjoining sessions of the stock exchange) for the actual and the wavelet denoised series by using a quantile shrinking method (for the 0.9 quantile). We observe, in particular, that the probability mass is more concentrated for the denoised series, as expected.

`[Figure 9 around here]`

[3] We computed the wavelet packet transforms after choosing the best basis with the Coifman-Wickerhauser algorithm. As it resulted in a large basis, we were not able to plot the wavelet packet coefficients.

3.2 Estimation of the fractal dimension

A common test for long run persistence in time series is done by estimating the fractional degree of integration of an ARFIMA model, which is related to the fractal dimension of the series (see Campbell et al. (1997, 59-60)). Next we present a Monte-Carlo estimation of the fractional parameter by using wavelet methods for denoising the series (see Jensen (1999) for an alternative wavelet based method).

The following method was used: first, by bootstrapping the original series, 100 samples were build; second, we applied a wavelet transformation and applied a maximum likelihood estimator on the transformed data, thus getting 100 samples for the fractional parameter; at last, a kernel was estimated on the estimated sample for the fractional parameter.

Figure 10 shows that the distribution is slightly biased towards a negative fractional value of the fractional parameter. A zero value would mean that the series is a random walk. Negative values are related to some long run memory of an integral form. This result reveals the presence of a non-linear fractionally integrated stochastic process.

[Figure 10 around here]

4 Final remarks

Given the exploratory nature of this paper, all the results should be taken merely as impressionistic. An outstanding feature of wavelets, which is not present in denoising by using spectral methods, is their location properties. This means that wavelets are particularly useful for singularity detection, or abrupt changes in the trend. Therefore, the trade-off between location in time and frequencies which we presented in most denoising exercises, seems to have a much more flexible solution when we use wavelet methods.

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Figure 1

Daubechies_4 scaling function: phi_4(t/2), phi_4(2t-2)

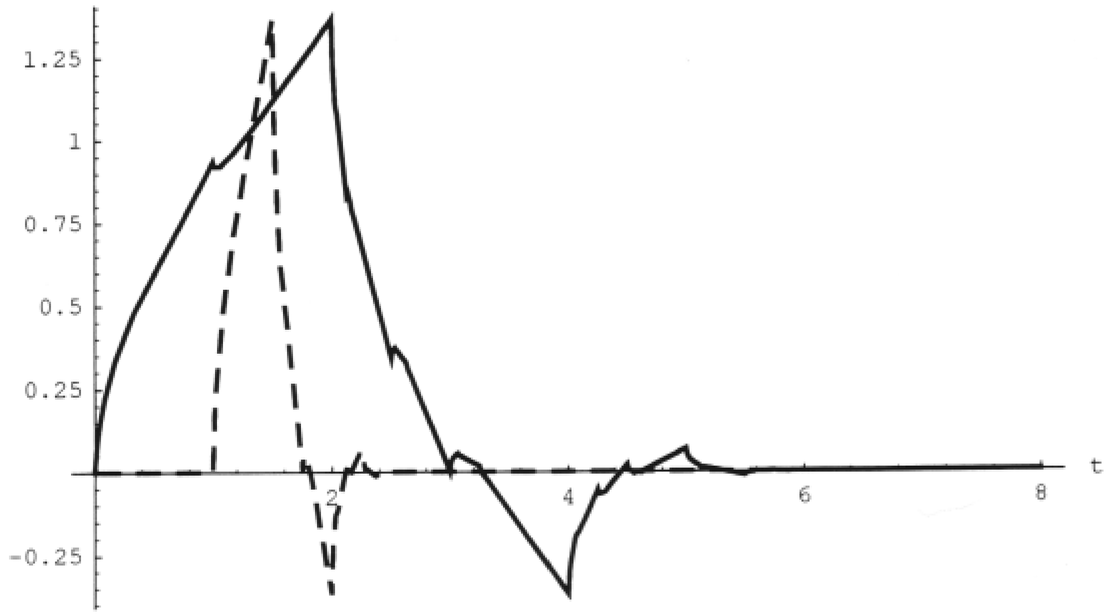


Figure 2

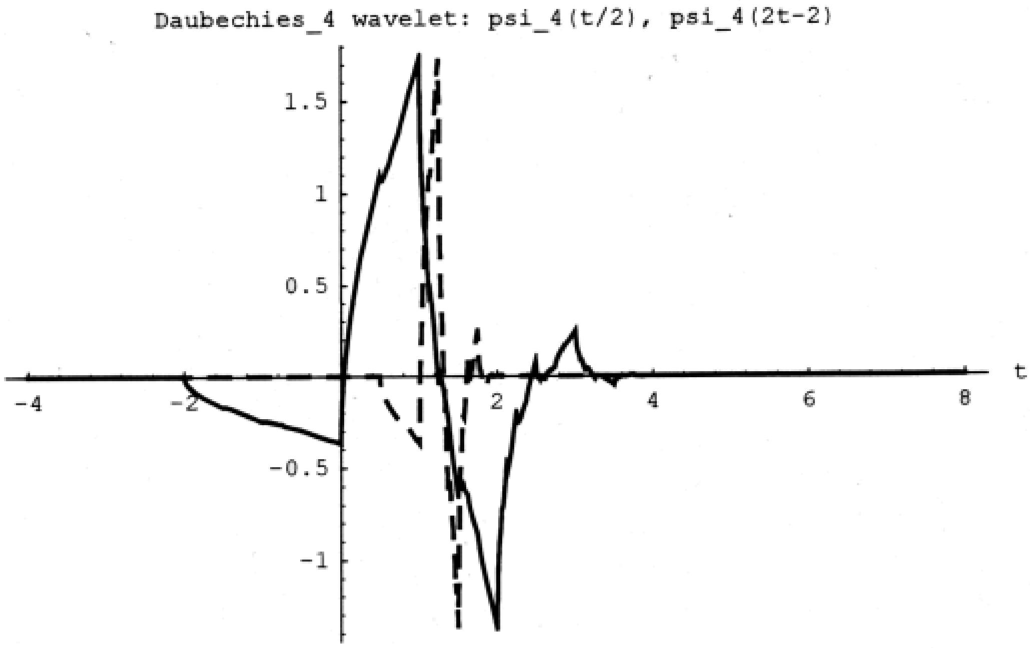
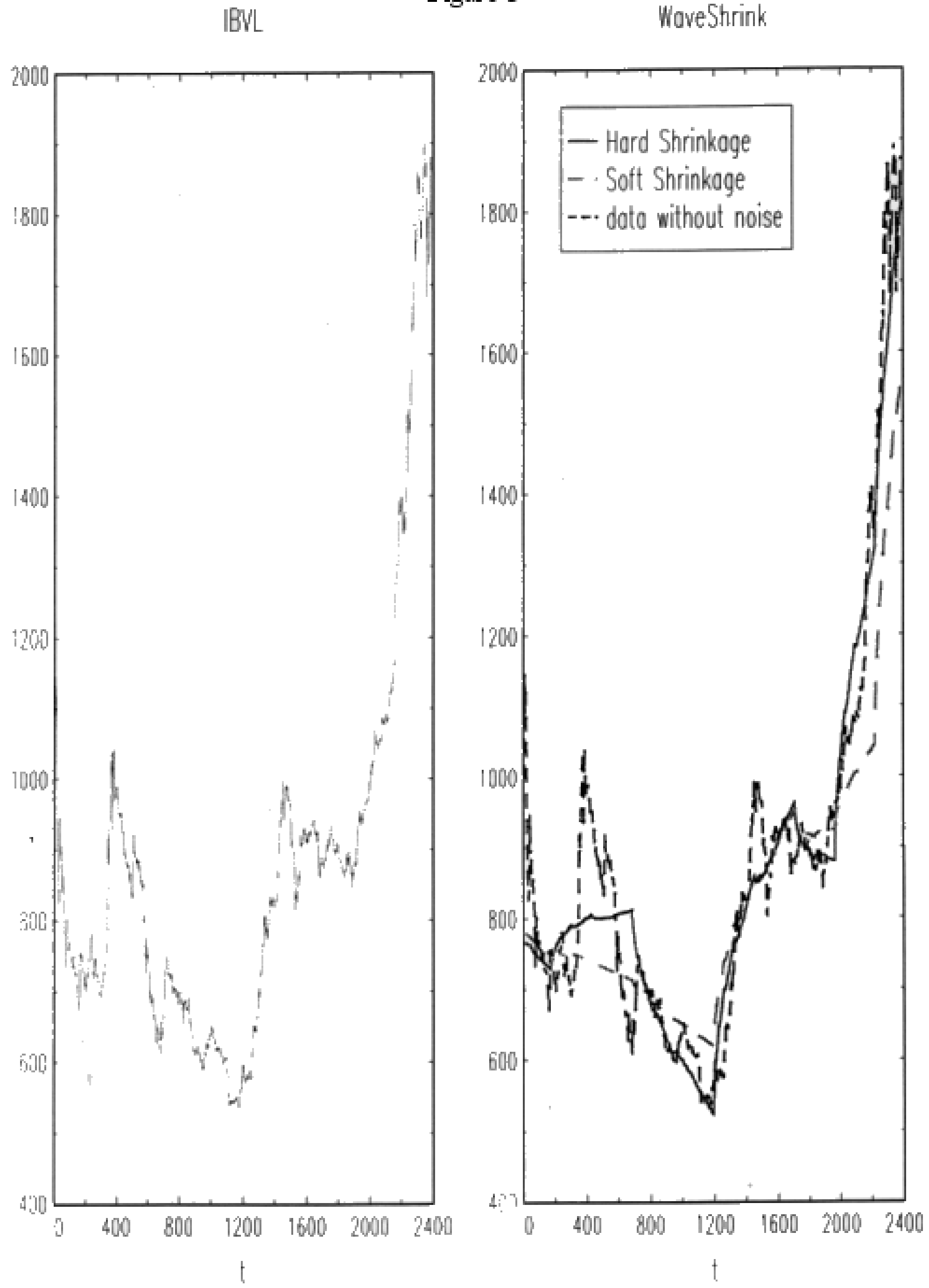
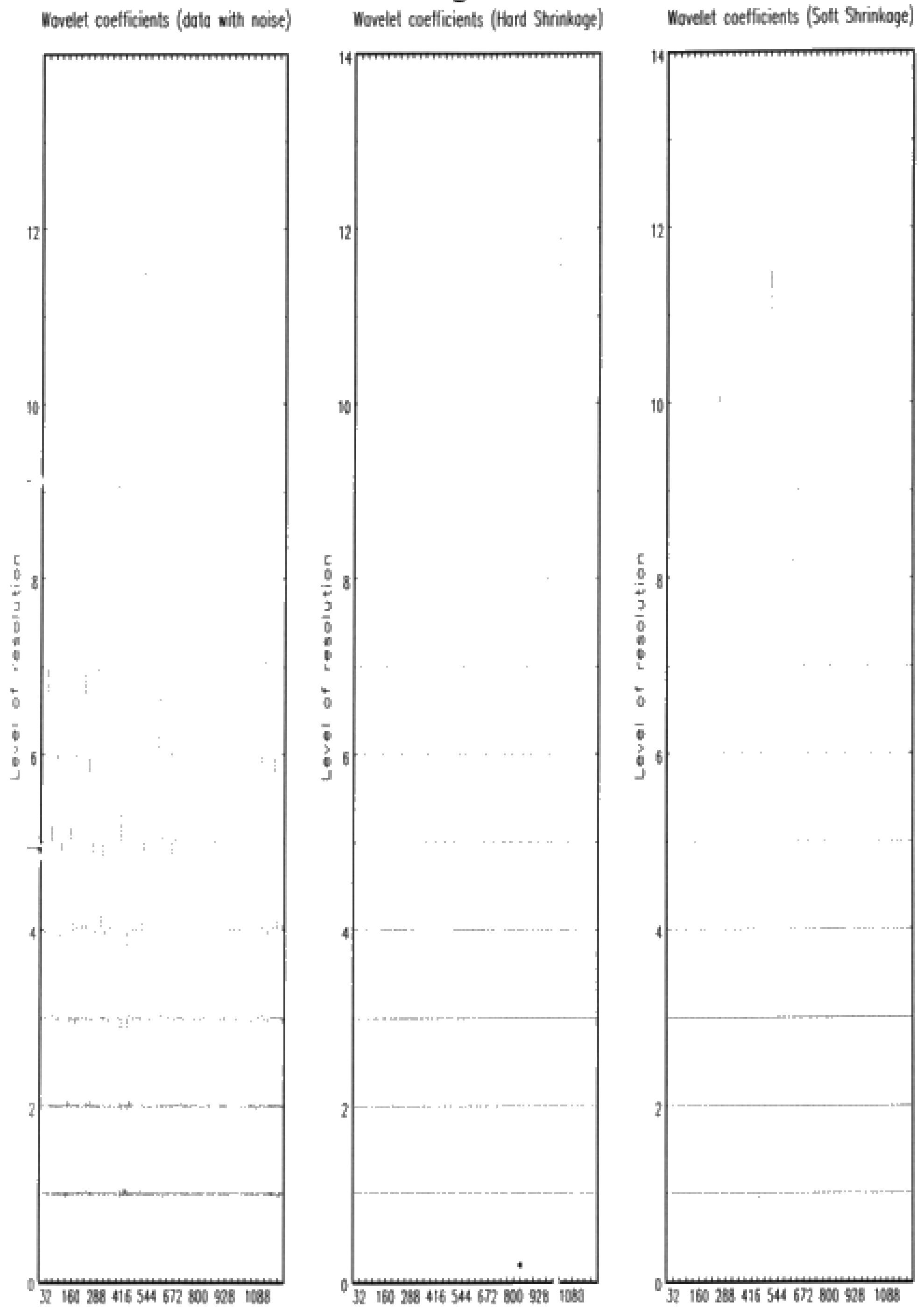


Figure 3



IBVL
WaveShrink
Hard Shrinkage
Soft Shrinkage
data without noise
2000
1800
1600
1400
1200
1000
800
600
400
0
400
800
1200
1600
2000
2400
t
t

Figure 4



Wavelet coefficients (data with noise)
Wavelet coefficients (Hard Shrinkage)
Wavelet coefficients (Soft Shrinkage)
Level of resolution
32 160 288 416 544 672 800 928 1088

Figure 5

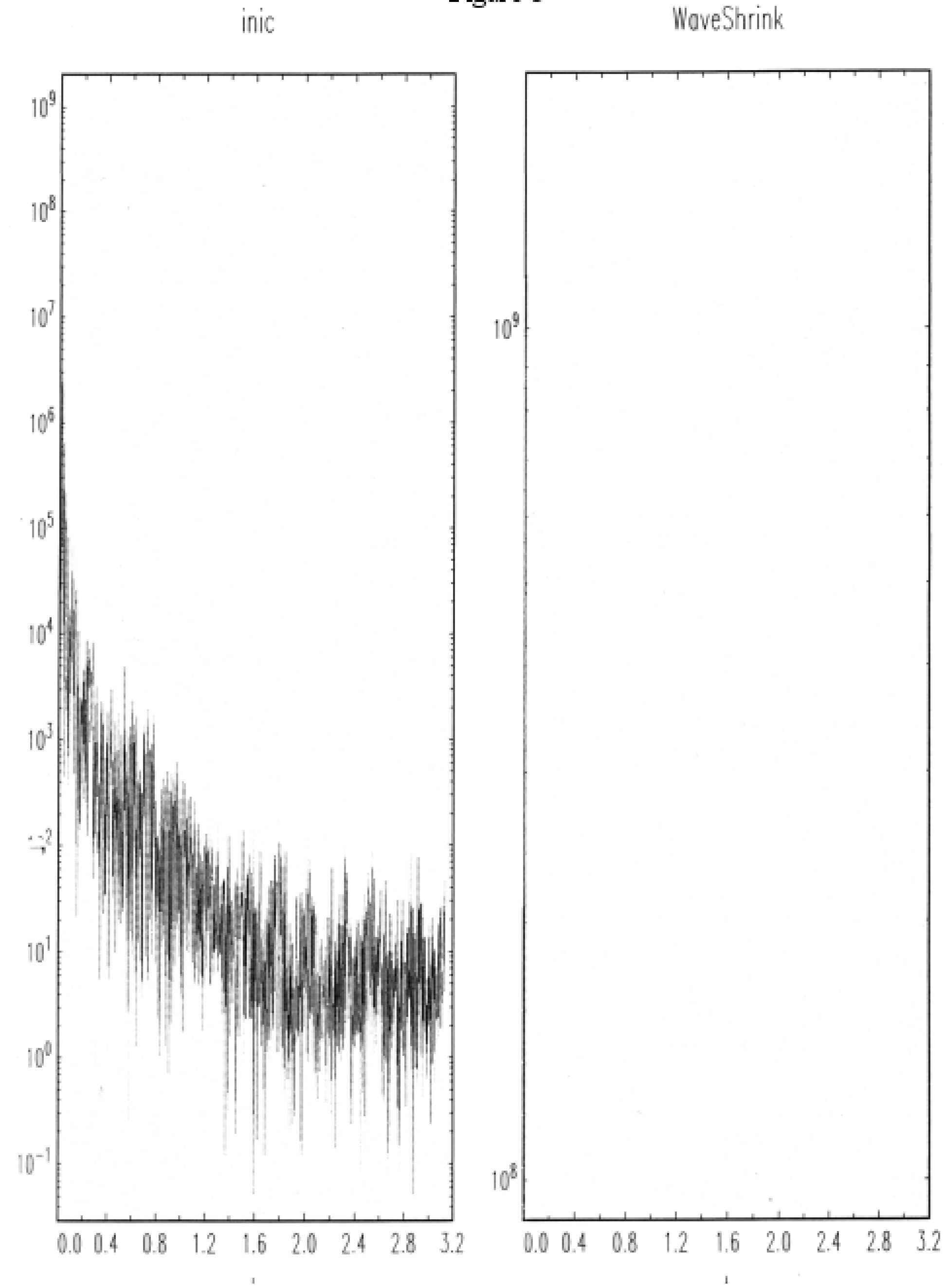
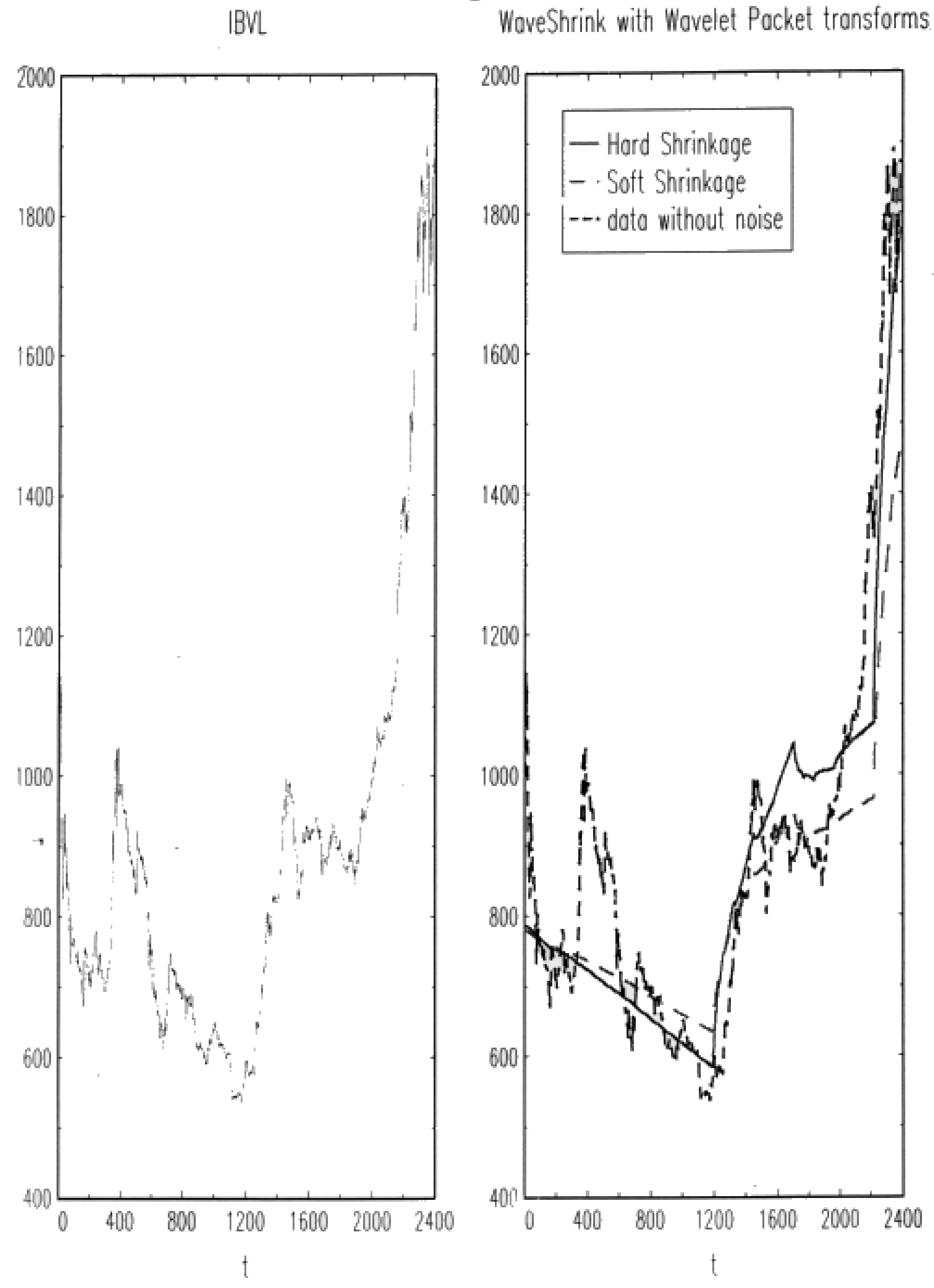

inic
WaveShrink
10^9
10^8
10^7
10^6
10^5
10^4
10^3
10^1
10^0
10^-1
0.0 0.4 0.8 1.2 1.6 2.0 2.4 2.8 3.2
10^9
10^8
0.0 0.4 0.8 1.2 1.6 2.0 2.4 2.8 3.2

Figure 6



IBVL
WaveShrink with Wavelet Packet transforms
Hard Shrinkage
Soft Shrinkage
data without noise
2000
1800
1600
1400
1200
1000
800
600
400
0
400
800
1200
1600
2000
2400
t

Figure 7

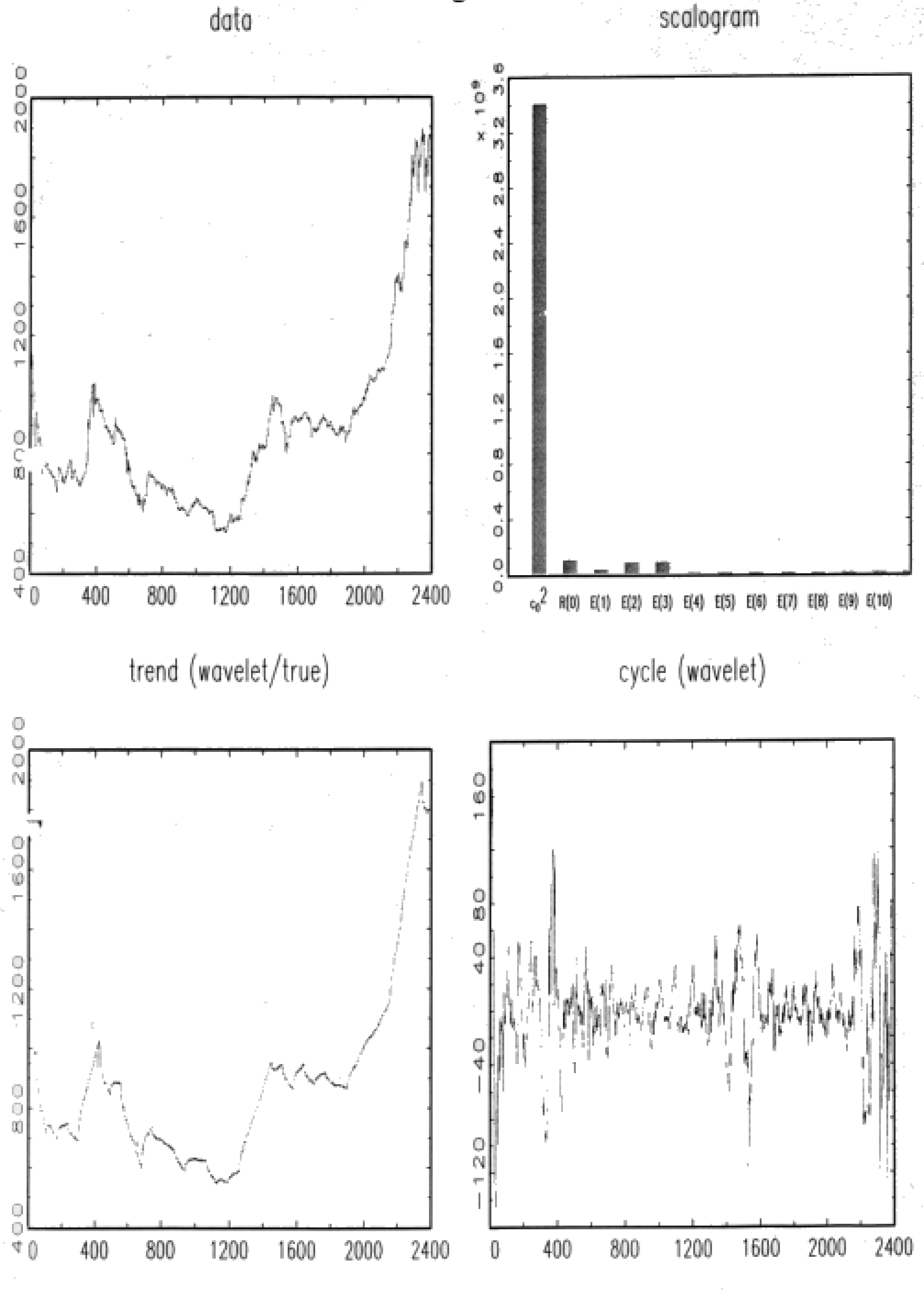

data
scalogram
trend (wavelet/true)
cycle (wavelet)
0 400 800 1200 1600 2000 2400
400 800 1200 1600 2000
x 10⁹
0 0.4 0.8 1.2 1.6 2.0 2.4 2.8 3.2 3.6
R(0) E(1) E(2) E(3) E(4) E(5) E(6) E(7) E(8) E(9) E(10)
-120 -40 40 80 160

Figure 8

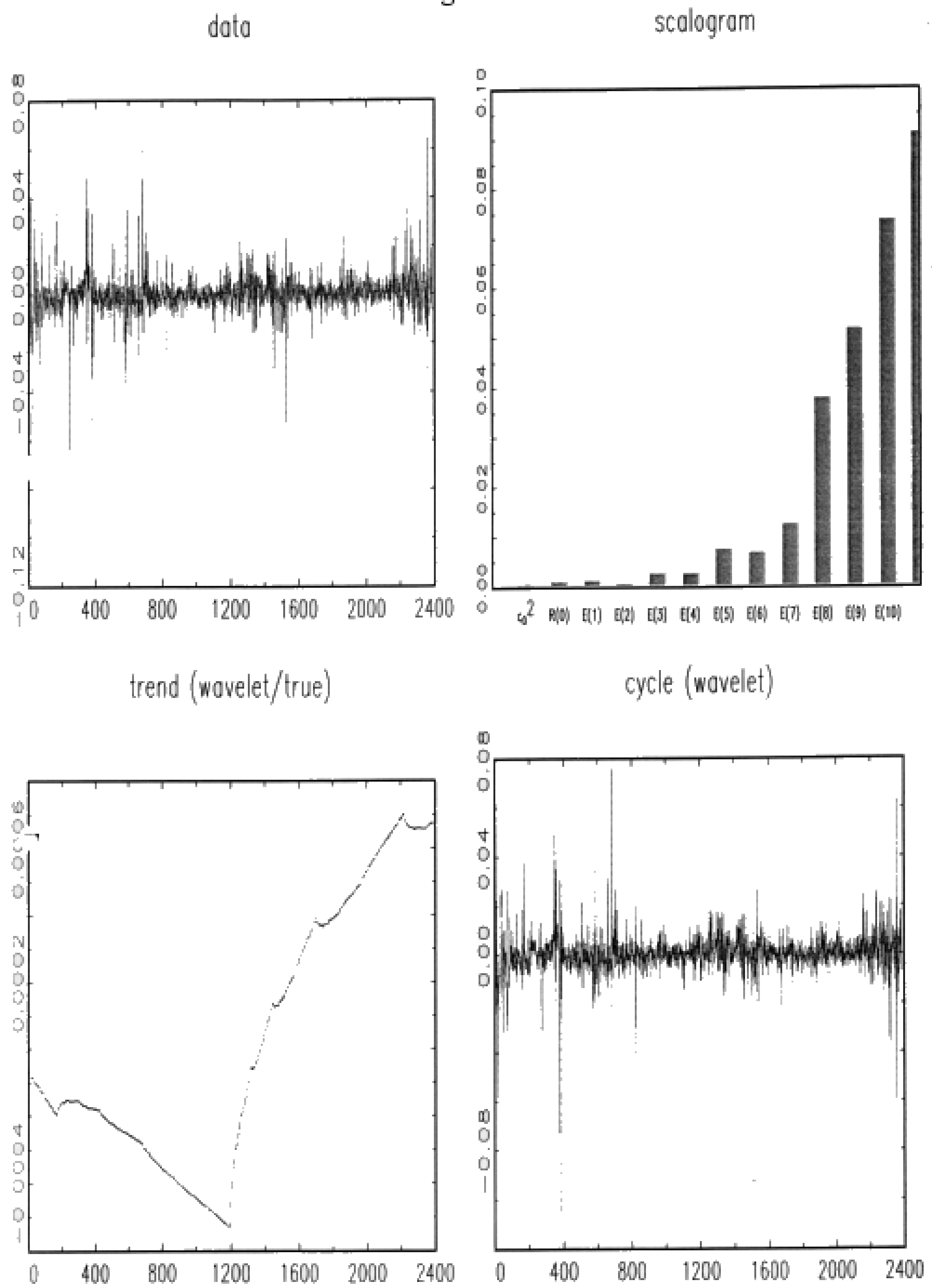
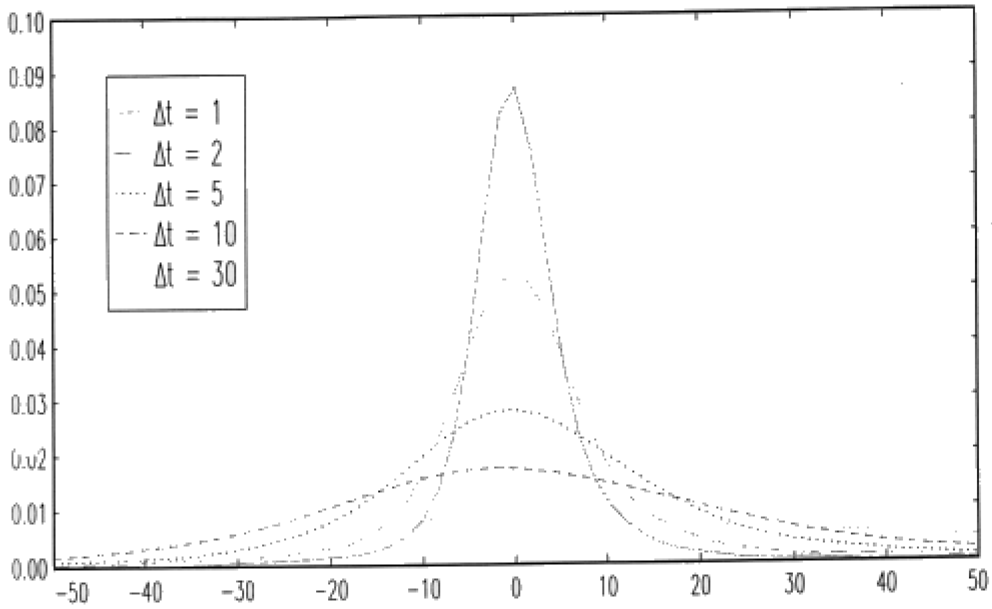

data
scalogram
trend (wavelet/true)
cycle (wavelet)
0 400 800 1200 1600 2000 2400
0.00 0.02 0.04 0.06 0.08 0.10
E(1) E(2) E(3) E(4) E(5) E(6) E(7) E(8) E(9) E(10)

Figure 9

Probability density of the return



Probability density of the return from the denoised series

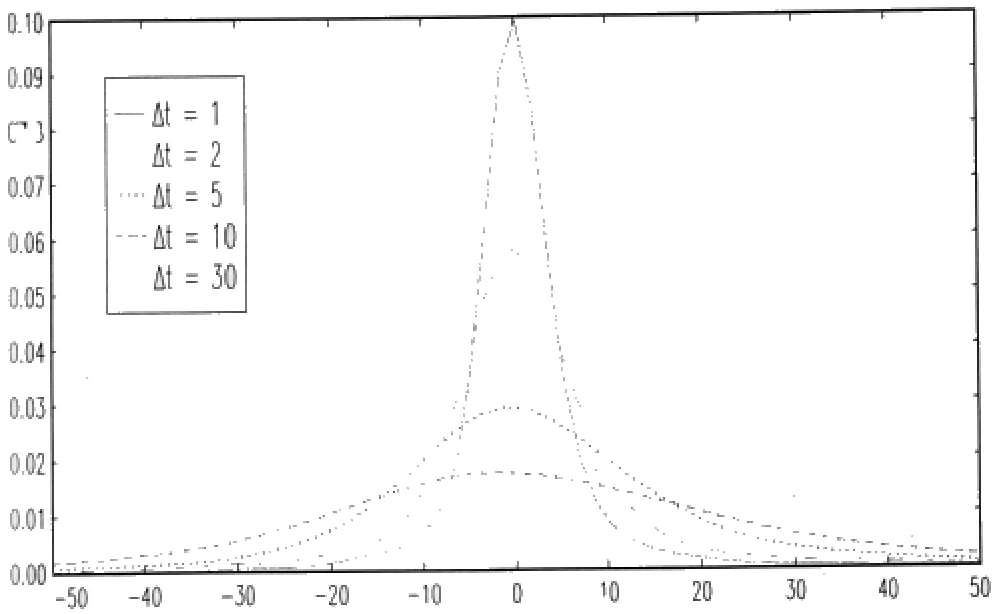
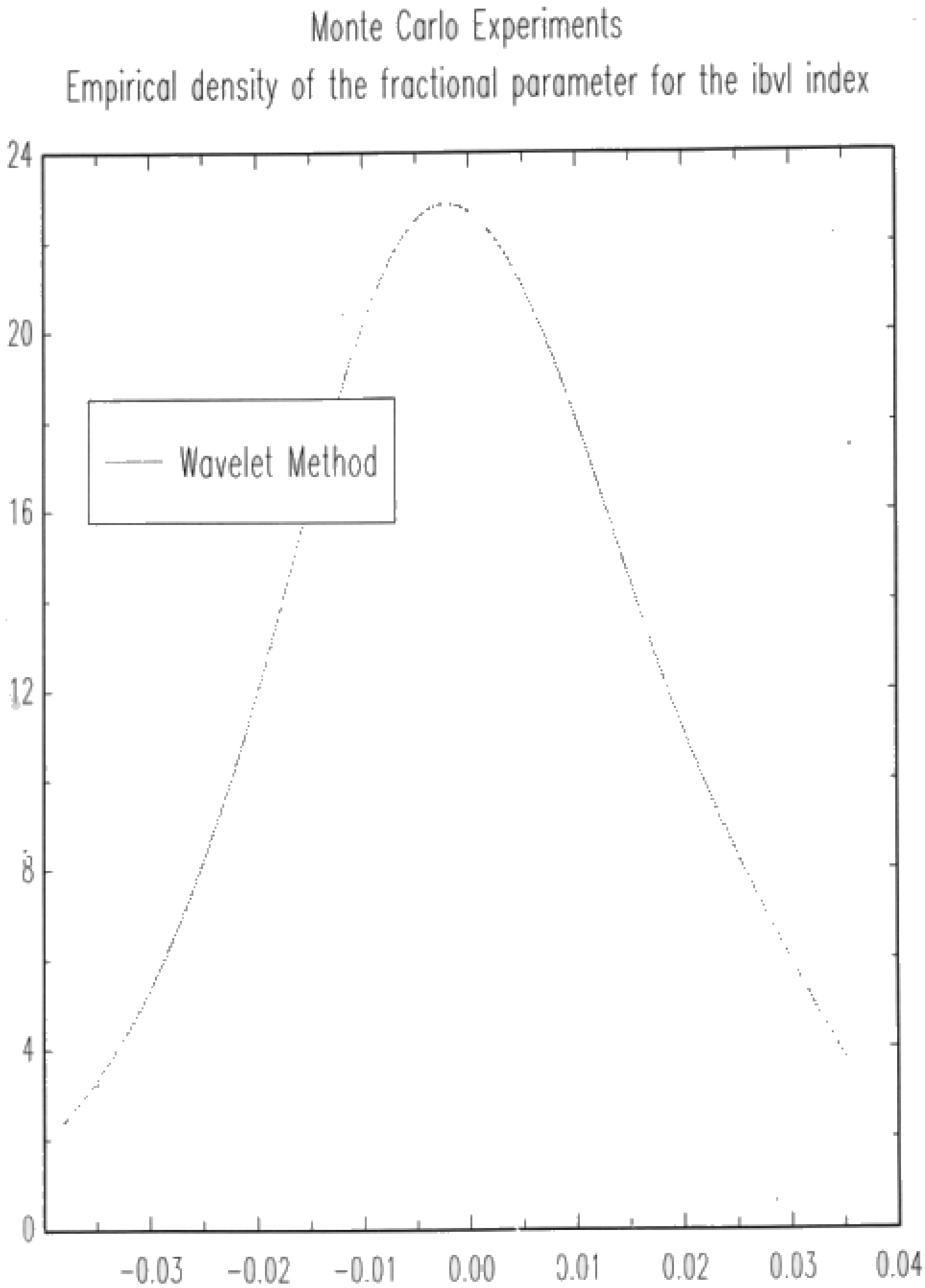


Figure 10


Monte Carlo Experiments
Empirical density of the fractional parameter for the ibvl index
Wavelet Method
24
20
16
12
8
4
0
-0.03
-0.02
-0.01
0.00
0.01
0.02
0.03
0.04