Non-smooth and singular dynamics and bifurcations in a Cournot-Ramsey model

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We study:

- a dynamic general equilibrium model;
- which extend the ubiquitous Ramsey (1928) model, with endogenous labour;
- for an imperfectly competitive (Cournotian) economy;
- where the markups, the number of Dixit-Stiglitz (1977) varieties, and the number of firms are endogenous;
- using a (very) limited toolbox;
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There are two uncommon (in economics) features in the model:

- there is a change in regime, from oligopoly to monopoly, when the number of firms is reduced to one : this generates a piecewise-smooth differential equation;
- labour and consumption can be complementary or substitutable: this generates a singular DAE.

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We were struggling with those problems for a long time, until we found:

- on the non-smooth dynamics: Di Bernardo, Champneys and all (2008) and Leine and Nijmeijer (2004).
- on the singular part: Venkatasubramanian et all (1995) and Riazza (2008).
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Summary:

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2. Local dynamics
3. Dynamics at the boundaries
4. Phase diagrams for some cases

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The model

Components of the model:

- a representative household who solves a standard intertemporal optimization problem of consumption, labour supply and saving
- a representative producer of final goods, which is price-taker;
- $z(t)$ intermediate sectors, each one having $n(v,t)$ firms, all operating in imperfectly competitive (Cournotian) markets: firms compete in quantities in the same industry and in prices with other industries;
- equilibrium equations for all markets.

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Determines

$$\max_{[C(t),L(t)]_{t\geq 0}} U = \int_0^{\infty} \left[\ln(C(t)) - \frac{\xi}{1+\tau} L(t)^{1+\tau} \right] e^{-\rho t} dt$$

subject to

$$\dot{K}(t) = w(t)L(t) + r(t)K(t) + \Pi(t) - C(t)$$

and

$$K(0) = K_0 > 0, \quad \lim_{t\to\infty} K(t) e^{-\overline{r}(t)t} \geq 0$$

exogenous variables: $w(t)$, $r(t)$, $\Pi(t)$;
endogenous variables: $C(t)$= consumption, $K(t)$= stock of capital, $L(t)$=labour effort;
parameters: $\rho > 0$, $\xi > 0$, $\tau > 0$

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Consumption Euler equation:

$$\frac{\dot{C}(t)}{C(t)} = r(t) - (\rho + \delta)$$

Consumption - labour supply arbitrage equation

$$L(t) = \left(\frac{w(t)}{\xi C(t)} \right)^{\frac{1}{\tau}}$$

Transversality condition

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$$\max_{[y(v,t)]_{v\in[0,z(t)]}} \pi(t) = P(t)Y(t) - \int_0^{z(t)} p(v,t)y(v,t)dv$$

subject to

$$Y(t) = z(t)^{\frac{1}{1-\sigma}} \left[\int_0^{z(t)} y(v,t)^{\frac{\sigma-1}{\sigma}} dv \right]^{\frac{\sigma}{\sigma-1}}$$

exogenous variables: $p(v,t)$, $P(t)$ prices;
endogenous variables: $y(v,t)$ input of variety $v \in [0, z(t)]$, $z(t) \in (0,1]$, $Y(t)$ output;
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Demand for input v

$$y(v,t) = \left[\frac{p(v,t)}{P(t)}\right]^{-\sigma} \frac{Y(t)}{z(t)}$$

where

$$P(t) = \left[\frac{1}{z(t)} \int_0^{z(t)} p(v,t)^{1-\sigma} \, dv\right]^{\frac{1}{1-\sigma}} = 1$$

The representative firm $i \in n(v,t)$ in industry $v \in [0, z(t)]$ determines

$$\max_{L_i(v,t),K_i(v,t)} \Pi_i(v,t) = p(v,t)\, y_i(v,t) - w(t)\, L_i(v,t) - r(t)\, K_i(v,t)$$

subject to

$$p(v,t) = \left[z(t) \frac{y_i(v,t) + \sum_{k \neq i \in V} y_k(v,t)}{Y(t)} \right]^{-\sigma}$$

and

$$y_i(v,t) = F\left(K_i(v,t) - \phi L(v,t)\right)$$

endogenous variables: $K_i(v,t)$, $L_i(v,t)$, $y_i(v,t)$, $p(v,t)$;
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Optimality conditions

$$[1-\mu_i(v,t)]\,F_{L,i}(v,t) = \frac{w(t)}{p(v,t)}$$
$$[1-\mu_i(v,t)]\,F_{K,i}(v,t) = \frac{r(t)}{p(v,t)},$$

where:

$$\mu_i(v,t) = [p(v,t) - MC_i(v,t)]/p(v,t) \in (0,1)$$

is the Lerner index for firm i in industry v

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$$MC_i(v,t) = w(t)/F_{L,i}(v,t) = r(t)/F_{K,i}(v,t)$$

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We consider a doubly symmetric equilibrium both at the intra-industrial and the inter-industrial levels, which implies

$$\mu_i(v,t) = \mu(v,t) = 1/[\sigma n(v,t)]$$

Aggregate output becomes

$$Y(t) = AF(K(t),L(t)) - z(t)n(t)\phi$$

Aggregation profits are

$$\Pi(t) = \int_0^{z(t)} \Pi(v,t)\,dv = \mu(t)F(K(t),L(t)) - z(t)n(t)\phi$$

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Assumption: there is instantaneous free entry (and zero entry costs), then $\Pi(t) = 0$.

Solving the indetermination of nz: two regimes exist

(1) a monopolistic-competition (MC) regime, when K is small:
$n = 1$ and entry determines the equilibrium value of $z \in (0,1)$.

(2) oligopoly regime, when K is larger:
$z = 1$ and entry determines the equilibrium value of $n \geq 1$.

Then

$$\mu = \mu(K,L) = \begin{cases} 1/\sigma & \text{if } m(K,L) \geq 1/\sigma \\ m(K,L) & \text{if } m(K,L) \leq 1/\sigma. \end{cases}$$

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The dynamic general equilibrium:

Is defined by the paths $\{(C(t), K(t), L(t), r(t), w(t), z(t), n(t),\ t \in \mathbf{R}_+\}$ that solve

$$\begin{aligned}
\dot{C} &= C(r(K,L) - \rho), \\
\dot{K} &= (r(K,L)K - \alpha C)/\alpha, \\
0 &= g(C,K,L) = C - r(K,L)K(L^*/L)^{1+\tau},
\end{aligned}$$

where

$$r(K,L) = \begin{cases} (1-1/\sigma)\alpha F(K,L)/K & \text{if } m(K,L) \geq 1/\sigma \\ (1-m(K,L))\alpha F(K,L)/K & \text{if } m(K,L) \leq 1/\sigma \end{cases}$$

and

$$m(K,L) = (\phi/\sigma)^{1/2} F(K,L)^{-1/2},\ F(K,L) = K^{\alpha} L^{1-\alpha},\ 0 < \alpha < 1$$

given

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and

$$
m(K,L) = (\phi/\sigma)^{1/2} F(K,L)^{-1/2},\ F(K,L) = K^{\alpha} L^{1-\alpha},\ 0 < \alpha < 1
$$

given

$$
K(0) = K_0 > 0,\ \lim_{t\to\infty} e^{-\rho t} K(t)/C(t) = 0
$$

Main surfaces

- Constraint surface

$$\mathscr{C} \equiv \{(C,K,L) \in \mathbf{R}^3_{++} : \; g(C,K,L,\theta) = 0\}.$$

- switching boundary

$$\Sigma \equiv \{(C,K,L) \in \mathscr{C} : \; m(K,L,\theta) = 1/\sigma\}.$$

splits $\mathscr{C}$ into $\mathscr{C}_1 \cup \Sigma \cup \mathscr{C}_2$

- locus for regular and smooth bifurcations

$$\mathscr{B} \equiv \{(C,K,L) \in \mathscr{C}_2 : \; m(C,K,L) = 1/\sigma_B \equiv \frac{2(1-\alpha)}{2-\alpha}\}$$

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$$\mathscr{S} \equiv \left\{ (C,K,L) \in \mathscr{C}_2 : \; m(C,K,L) = 1/\sigma_S \equiv \frac{2(\alpha+\tau)}{1+2\tau+\alpha} \right\}.$$

  where $g_L(C,L,K) = 0$
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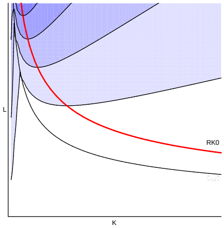
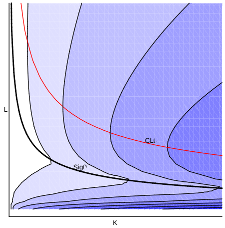
The main functions: contour plots

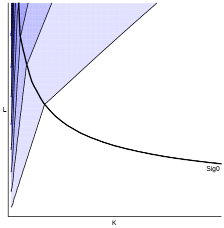
$r(K,L)$ function
$\sigma < \sigma_S(\alpha,\tau)$



$C(K,L)$ function
$\sigma < \sigma_B(\alpha)$



$\sigma > \sigma_S(\alpha,\tau)$



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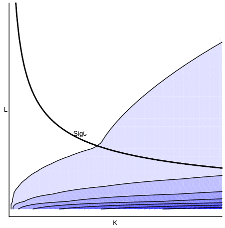
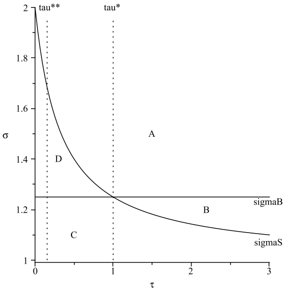


A first bifurcation diagram



- Existence of singularities

  if $\sigma > \sigma_S$ then there is no singularity for function g, $\mathscr{S}$ is empty;

  if $\sigma = \sigma_S$ then there is a singularity singularities for function g and $\mathscr{S} = \Sigma$;

  if $1 < \sigma < \sigma_S$ then function g has a singularity

- Number of stationary equilibria (steady states)

$$L^* = 1, \; C^* = \frac{\rho}{\alpha} K^*, \; K^* = \{K > 0 : r(K, L^*) = \rho\}$$

  If $\sigma > \sigma_B$ there is a single equilibrium;

  If $\sigma \geq \sigma_B$ there may be one, two or three equilibria

Let $(C,K,L) \notin \mathscr{S}$, then the original DAE can be reduced to an ODE

$$\begin{aligned} \dot{C} &= C\left(R(C,K)-\rho\right) \\ \dot{K} &= \alpha^{-1}R(C,K)K - C \end{aligned}$$

where

$$R(C,K) = \begin{cases} (1-1/\sigma)F_K(K,L_1(C,K)) & \text{if } C \geq C_\Sigma(K) \\ (1-m(K,L_2(C,K))F_K(K,L_2(C,K)) & \text{if } C < C_\Sigma(K) \end{cases}$$

where $L_j(C,K) := \{L : g_j(C,K,L) = 0\}$ is implicitly defined for $j = 2$

Local dynamics

The Jacobian:

- if $(C^*, K^*, L^*) \in \mathscr{C}_1$, has
$$\mathrm{tr}(J_1) = \rho > 0, \det(J_1) = -\rho^2(1+\tau^*)(1+\tau)/(\alpha+\tau) < 0$$
- for $(C^*, K^*, L^*) \in \mathscr{C}_2$
$$\begin{aligned}\mathrm{tr}(J_2) &= \frac{\rho m_S(2-m^*)}{2(m_S-m^*)}\\ \det(J_2) &= -\frac{\rho^2(1+\tau)(1+\tau^*)m_S(m_B-m^*)}{(\alpha+\tau)m_B(m_S-m^*)}.\end{aligned}$$
regular saddle-node bifurcation: $\det(J_2) = 0$ if $m^* = m_B$
- for $(C^*, K^*, L^*) \in \Sigma$ the generalized Jacobian ∂J has
$$\begin{aligned}\mathrm{tr}(J_\Sigma) &= \{q\mathrm{tr}(J_1) + (1-q)\mathrm{tr}(J_2),\ 0 \le q \le 1\}\\ \det(J_\Sigma) &= \{q\det(J_1) + (1-q)\det(J_2),\ 0 \le q \le 1\}\end{aligned}$$
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Let $(C,K,L) \in \mathscr{S}$, then consider the transformed vector field

$$Z^T(C,K,L) = \begin{pmatrix} (g_{2,L}\dot{C})(C,L,K) \\ (g_{2,L}\dot{K})(C,L,K) \\ -(g_{2,C}\dot{C} + g_{2,K}\dot{K})(C,K,L) \end{pmatrix}$$

If $\sigma < \sigma_S$ and $\sigma_B \neq \sigma_S$, and $\phi = \phi_S$, then the equilibrium point $(K^*_{\mathscr{S}}, C^*_{\mathscr{S}}, L^*) \in \mathscr{S}$ undergoes a singularity induced bifurcation, in the sense of Venkatasubramanian et all (1995)):

$$g_L(C,K,L) = 0$$

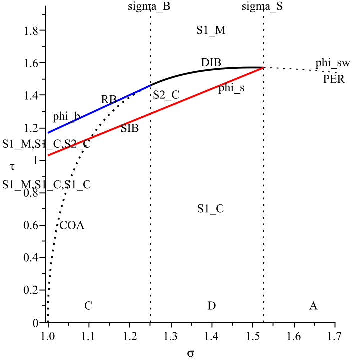
$$\mathrm{tr}\begin{pmatrix} \frac{\partial \dot{C}}{\partial C} g_C & \frac{\partial \dot{C}}{\partial C} g_K \\ \frac{\partial \dot{K}}{\partial C} g_C & \frac{\partial \dot{K}}{\partial C} g_K \end{pmatrix} \neq 0$$

$$\det\begin{pmatrix} \frac{\partial \dot{C}}{\partial C} & \frac{\partial \dot{C}}{\partial K} & \frac{\partial \dot{C}}{\partial L} \\ \frac{\partial \dot{K}}{\partial C} & \frac{\partial \dot{K}}{\partial K} & \frac{\partial \dot{K}}{\partial L} \\ g_C & g_K & g_L \end{pmatrix} \neq 0$$

$$\det\begin{pmatrix} \frac{\partial \dot{C}}{\partial C} & \frac{\partial \dot{C}}{\partial K} & \frac{\partial \dot{C}}{\partial L} & \frac{\partial \dot{C}}{\partial \phi} \\ \frac{\partial \dot{K}}{\partial C} & \frac{\partial \dot{K}}{\partial K} & \frac{\partial \dot{K}}{\partial L} & \frac{\partial \dot{K}}{\partial \phi} \\ g_C & g_K & g_L & g_\phi \\ g_{LC} & g_{LK} & g_{LL} & g_{L\phi} \end{pmatrix} \neq 0$$

Local dynamics: a second bifurcation diagram

if $\tau < (1-\alpha)/\alpha$



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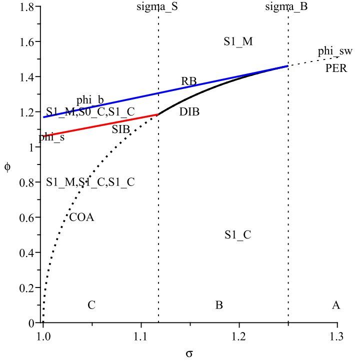

RB = regular bifurcation, DIB = discontinuity induced bifurcation, SIB = singularity induced bifurcation, PER = persistence of equilibria, COA= coalescence of equilibria

Dynamics at the switching boundary

We consider the solution flow $\Phi(t) = (\Phi_C(t), \Phi_K(t), \Phi_L(t))$ and $\Phi_1(t)$ and $\Phi_2(t)$, on both sides of Σ, for $(C^*, K^*, L^*) \notin \Sigma$

the flow $(\Phi_{1K}(t), \Phi_{1L}(t))$ is the solution of

$$\begin{aligned} \dot{K} &= r_1(K,L)\left(1 - \left(\frac{L^*}{L}\right)^{1+\tau}\right)(K/\alpha) \\ \dot{L} &= \left(\rho - r_1(K,L)\left(\frac{L^*}{L}\right)^{1+\tau}\right) L/(\alpha+\tau) \end{aligned}$$

and the flow $(\Phi_{2K}(t), \Phi_{2L}(t))$ is the solution of

$$\begin{aligned} \dot{K} &= r_2(K,L)\left(1 - \left(\frac{L^*}{L}\right)^{1+\tau}\right)(K/\alpha), \\ \dot{L} &= \left[\rho - r_2(K,L)\left(\frac{L^*}{L}\right)^{1+\tau} + \frac{m(K,L) r_2(K,L)}{2(1-m(K,L))}\left(1 - \left(\frac{L^*}{L}\right)^{1+\tau}\right)\right] \frac{2(1-m(K,L)}{(1+\alpha+2\tau)(m_S - r} \end{aligned}$$

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Let $h(K,L) = m(K,L) - 1/\sigma$ (then $(K,L) \in \Sigma \Leftrightarrow h(K,L) = 0$) and define

$$v_j = \frac{\partial h}{\partial K}\frac{d\Phi_{j,K}(t)}{dt} + \frac{\partial h}{\partial L}\frac{d\Phi_{j,L}(t)}{dt},\ j = 1,2$$

- If we evaluate at Σ, we get

$$v_2 = v_1 \frac{2(1-1\sigma)(\alpha+\tau)}{(1+\alpha+2\tau)(m_S - 1/\sigma)}.$$

  where v_1 is a function of the parameters

- There are three types of interactions of $\Phi(t)$ with Σ
  1. $\Phi(t)$ is tangent to Σ if $v_1 = v_2 = 0$ and, evaluated at Σ

$$\left.\frac{dL}{dK}\right|_{h=0} = \frac{d\Phi_{1L}(t)}{d\Phi_{1K}(t)} = \frac{d\Phi_{2L}(t)}{d\Phi_{2K}(t)}$$

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  2. $\Phi(t)$ is transverse to Σ if $\sigma > \sigma_S$ $(v_1 v_2 > 0)$
  3. $\Phi(t)$ slides along to Σ if $\sigma < \sigma_S$ $(\text{sign}(v_1) \neq \text{sign}(v_2))$

Let $h(K,L) = m(K,L) - 1/\sigma$ (then $(K,L) \in \Sigma \Leftrightarrow h(K,L) = 0$) and define

$$v_j = \frac{\partial h}{\partial K}\frac{d\Phi_{j,K}(t)}{dt} + \frac{\partial h}{\partial L}\frac{d\Phi_{j,L}(t)}{dt}, \; j = 1,2$$

- If we evaluate at Σ, we get

$$v_2 = v_1 \frac{2(1-1\sigma)(\alpha+\tau)}{(1+\alpha+2\tau)(m_S - 1/\sigma)}.$$

  where v_1 is a function of the parameters
- There are three types of interactions of $\Phi(t)$ with Σ
  1. $\Phi(t)$ is tangent to Σ if $v_1 = v_2 = 0$ and, evaluated at Σ

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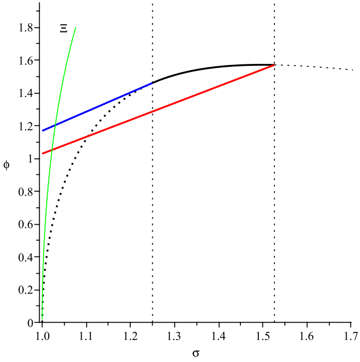
Dynamics at the switching surface



Let $\tau < \tau^*$ and

$$\Xi \equiv \rho(1-\alpha) + (\alpha+\tau)z(L_T/L^*)^{1+\tau^*} - (1+\tau)z(L_T/L^*)^{\tau^*-\tau}$$

If $\Xi < 0$ (below surface $\Xi = 0$) there are two tangent points between $\Phi(t)$ and Σ, if $\Xi = 0$ there is one, if $\Xi > 0$ there are no tangent points.

Consider again the transformed vector field $Z^T(C, K, L)$ and assume that $(C^*, K^*, L^*) \notin \mathscr{S}$. Three types of contact can exist:

- pseudo-equilibria if Z^T has a fixed point, (C_Ψ, K_Ψ, L_Ψ), on $\mathscr{S}$ which is not a fixed point of the original vector field: $g_2(C, K, L) = g_{2L}(C, K, L) = g_{2,C}\dot{C} + g_{2,K}\dot{K} = 0$.
- semi-equilibria: tangency point between the flow $\Phi(t)$ and $\mathscr{S}$ $g_2(C, K, L) = g_{2L}(C, K, L) = g_{2,LL} = 0$
- transverse singular points: all the other points. Maybe transverse sinks or sources (also called impasse points)

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Dynamics at the singular surface

Consider again the transformed vector field $Z^T(C, K, L)$ and assume that $(C^*, K^*, L^*) \notin \mathscr{S}$. Three types of contact can exist:

- pseudo-equilibria if Z^T has a fixed point, (C_Ψ, K_Ψ, L_Ψ), on $\mathscr{S}$ which is not a fixed point of the original vector field: $g_2(C, K, L) = g_{2L}(C, K, L) = g_{2,C}\dot{C} + g_{2,K}\dot{K} = 0$.
- semi-equilibria: tangency point between the flow $\Phi(t)$ and $\mathscr{S}$ $g_2(C, K, L) = g_{2L}(C, K, L) = g_{2,LL} = 0$
- transverse singular points: all the other points. Maybe transverse sinks or sources (also called impasse points)

In our case:

- pseudo-equilibria exist only if $\sigma < \sigma_S$ (and $\phi \neq \phi_S$): if $\tau > \tau^*$ they are unique , and if $\tau < \tau^*$ there are zero, one or two (only if $\phi_B < \phi < \phi_S$)
- If a pseudo-equilibria exists then if $L_\Psi < L^*$ it is a saddle-point and if $L_\Psi > L^*$ it is a source;
- there are no semi-equilibria

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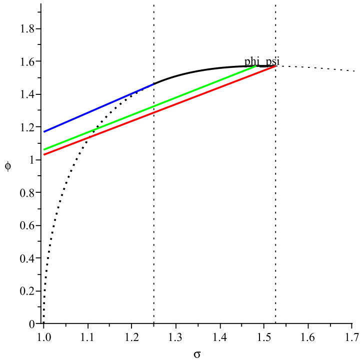
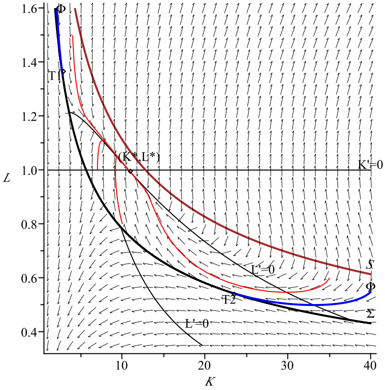
Bifurcation diagram

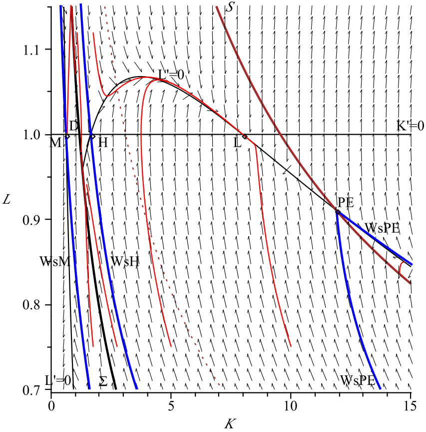


Selected phase diagrams



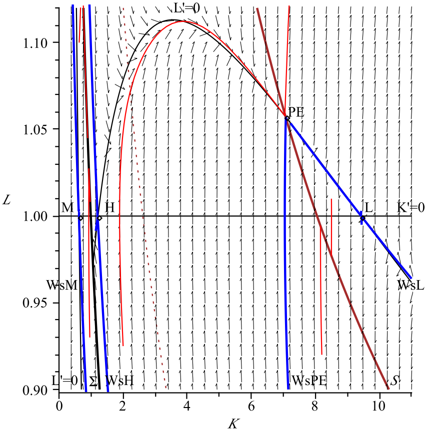
If $\tau < \tau^*$, $\sigma_B < \sigma < \sigma_S$, $\phi_S < \phi < \phi_B$

Selected phase diagrams



$\tau < \tau^*$, $\sigma < \sigma_B < \sigma_S$, $\phi_S < \phi < \phi_B$

Selected phase diagrams



$\tau < \tau^*$, $\sigma < \sigma_B < \sigma_S$, $\phi_\Sigma < \phi < \phi_S$

Thank you !